§31 Taylor's Theorem

Def Here and below let I be an open interval. It may be bdd or unbounded. Let $f: I \to \mathbb{R}$. Let $c \in I$. $f^{(k)}(c)$ means the k-th derivative at c; $f^{(0)} = f$.

Suppose $f^{(k)}(c)$ exists for $0 \le k \le n$. Then the n-th order Taylor polynomial of f at c is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

If $f^{(k)}(c)$ exists for all $k \ge 0$ then the Taylor series of f at c is

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Note: The radius of convergence could be zero, or infinity or any other value in $(0, \infty)$. Even if it converges it might not converge to $f(x)$.

For $n \ge 1$ we define the n-th remainder to be

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Clearly, $T(x) = f(x)$ iff $\lim_{n\to\infty} R_n(x) = 0$.

31.3 (Taylor's Theorem, version 1)

Let $f: I \to \mathbb{R}$. Suppose $f^{(n)}(x)$ exists $\forall\, x \in I$.
Then for all $x \in I - \{c\}$, (ie $x \neq c$), $\exists\, y$ between c and x s.t.

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n.$$

Pf Fix an $x \neq c$ in I and an $n \geq 1$. Then $\exists!\, M \in \mathbb{R}$ s.t. (unique)

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{M(x-c)^n}{n!}.$$

We will find a value y in between c and x s.t.

$$f^{(n)}(y) = M.$$

Let $g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(t-c)^k + \frac{M(t-c)^n}{n!} - f(t)$.

Claim: $g(c) = 0$. Pf: $g(c) = f(c) + \sum_{k=1}^{n-1} 0 + 0 - f(c) = 0$.

The first term of the sum is not zero!

Claim: $g^{(k)}(c) = 0$ for $k = 1, 2, \ldots, n-1$.

Pf $$g'(t) = \sum_{k=1}^{n-1} \frac{f^{(k)}(c)}{(k-1)!}(t-c)^{k-1} + \frac{M(t-c)^{n-1}}{(n-1)!} - f'(t).$$

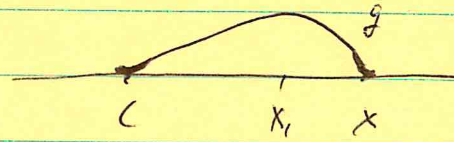
$$g'(c) = f^{(1)}(c) - f'(c) = 0$$

$$g''(t) = \sum_{k=2}^{n-1} \frac{f^{(k)}(c)}{(k-2)!}(t-c)^{k-2} + \frac{M(t-c)^{n-2}}{(n-2)!} - f''(t)$$

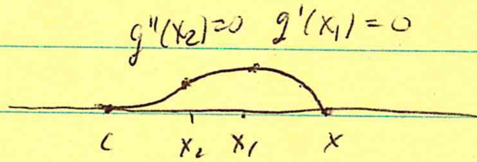
$$g''(c) = f^{(2)}(c) - f''(c) = 0.$$ Etc.

From the definition of g we know $g(x)=0$.

Thus $\exists\ x_1$ between c and x s.t. $g'(x_1)=0$. (Rolle's Thm)



$\exists\ x_2$ between c and x_1 s.t. $g''(x_2)=0$. (R's Thm)
(since $g'(c)=g'(x_1)=0$.)



$\exists\ x_3$ between c and x_2 s.t. $g'''(x_3)=0$.

Continue until we find x_n between c and x_{n-1} s.t. $g^{(n)}(x_n)=0$.

Let $y=x_n$. Now, remember what we are trying to prove, $f^n(y)=M$. Now

$$g^{(n)}(t) = 0 + M - f^{(n)}(t). \quad \text{for all } t\in I.$$

Let $t=y$. Then $g^n(y)=0=M-f^{(n)}(y) \Rightarrow M=f^{(n)}(y)$.

Now $R_n(x) = \dfrac{f^{(n)}(y)}{n!}(x-c)^n$. $\square$

31.4 Corollary Let $f: I \to \mathbb{R}$ be infinitely differentiable on I. Assume all the derivatives are bdd by a constant C. That is

$$|f^{(n)}(x)| \leq C \quad \forall x \in I, \ n = 0, 1, 2, 3, \ldots$$

Then

$$\lim_{n \to \infty} R_n(x) = 0 \quad \forall x \in I.$$

Pf Let $x \in I$. By the last thm

$$|R_n(x)| \leq \frac{C}{n!} |x-c|^n \quad \forall n \in \mathbb{N}.$$

The $\displaystyle\lim_{n \to \infty} \frac{|x-c|^n}{n!} = 0$ since by the Ratio Test the series $\displaystyle\sum_{n=0}^{\infty} \frac{|x-c|^n}{n!}$ converges, so the n-th term test requires $\displaystyle\lim_{n \to \infty} \frac{|x-c|^n}{n!} = 0$. $\square$

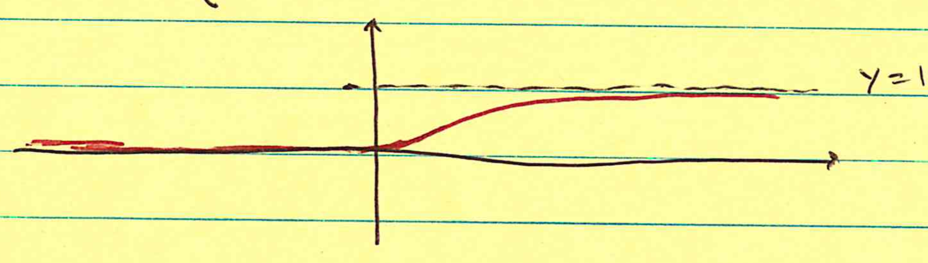
Ex 3 ← in textbook ~~$\cdot$~~ This is an example of a function that has all derivatives existing on $\mathbb{R}$, but whose Taylor series fails to converge to it. Functions that ~~have~~ = their Taylor series are called <u>analytic functions</u>. This examples shows that while all analytic functions and infinitely diff'able, the reverse is false.

The notation is $C^\infty(I)$ = all infinitely diff. func's on I.

omega → $C^\omega(I)$ = all analytic func's on I.

$$C^\omega(I) \subset C^\infty(I), \quad C^\omega(I) \neq C^\infty(I).$$

Let $f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0, \\ 0 & x \le 0. \end{cases}$



It is clear $f^{(n)}(x)$ exists for $x \neq 0$.
We will show $f^{(n)}(0) = 0$ for all n. Hence the Taylor Series at $x=0$ has all terms $= 0$. Thus, it converges to the zero function $\forall x \in \mathbb{R}$. So, it cannot converge to $f(x)$ on any open interval containing 0.

<u>Claim:</u> $f^{(n)}(x) = \begin{cases} e^{-\frac{1}{x}} p_n\left(\frac{1}{x}\right) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$

where $p_n(x)$ is a polynomial.

<u>Pf</u> This is obvious for $x<0$. For $x>0$ will use induction. The case $n=0$ is imediate. Suppose

$$f^{(n)}(x) = e^{-\frac{1}{x}} p\left(\tfrac{1}{x}\right) \quad \text{for some poly. } p\left(\tfrac{1}{x}\right).$$

$$f^{(n+1)}(x) = \left(e^{-\frac{1}{x}}\right)' p\left(\tfrac{1}{x}\right) + e^{-\frac{1}{x}}\left[p\left(\tfrac{1}{x}\right)\right]'$$

$$= e^{-\frac{1}{x}}\left(\tfrac{1}{x^2}\right) p\left(\tfrac{1}{x}\right) + e^{-\frac{1}{x}} p'\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x^2}\right)$$

$$= e^{-\frac{1}{x}}\left(\tfrac{1}{x^2}\right)\left(p\left(\tfrac{1}{x}\right) + p'\left(\tfrac{1}{x}\right)\right).$$

Since $\frac{1}{x^2}\left(p\left(\frac{1}{x}\right)+p'\left(\frac{1}{x}\right)\right)$ is a poly in $\frac{1}{x}$, we are done.

Now we have to check $x=0$. We will use the following fact that <u>you</u> can prove using L'Hopital's Rule:

$$\lim_{y\to\infty} y^k e^{-y} = 0 \quad \forall\, k \in \mathbb{N}.$$

Let $n=1$. $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{1}{x} f(x)$.

The limit as $x \to 0^-$ is zero since $f(x) = 0$ for $x<0$.

The limit as $x \to 0^+$ is $\lim_{x\to 0^+} \frac{1}{x} e^{-\frac{1}{x}} = \lim_{x\to\infty} y e^{-y} = 0$.

Thus $f'(0)=0$.

Assume $f^{(n)}(0)=0$. Then

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x-0} = \lim_{x\to 0} \frac{1}{x} f^{(n)}(x).$$

Again the limit as $x\to 0^-$ is zero.

$$\lim_{x\to 0^+} \frac{1}{x} f^{(n)}(x) = \lim_{x\to 0^+} \frac{1}{x} e^{-\frac{1}{x}} p_n\left(\frac{1}{x}\right)$$

$$= \lim_{y\to\infty} y p_n(y) e^{-y} = 0. \qquad \square$$