Section 31: Part 2, Bionomial Series

31.5 (Taylor's Thm, version 2: integral version)
Let $f: I \to \mathbb{R}$ and assume $f^{(n)}(x)$ exists for $x \in I$. Let $c \in I$. Then $\forall\, x \in I$

$$R_n(x) = \frac{\int_c^x (x-t)^{n-1} f^{(n)}(t)\, dt}{(n-1)!} \qquad (*)$$

Pf The proof uses the Fundamental Thm of Calc (34.1) and integration by parts (34.2) that we will cover later. We assume them for now.

For $n=1$ the result $(*)$ reduces to

$$R_1(x) = \int_c^x f^{(1)}(t)\, dt$$

By the FTC $\int_c^x f'(t)\, dt = f(x) - f(c)$.

But if you look back at the definition of $R_n(x)$ on page 250

$$R_1(x) = f(x) - \sum_{k=0}^{0} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= f(x) - f(c).$$

Now, assume $(*)$ holds for some fixed $n \geq 1$.

Recall the int. by parts formula $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$.

From the definition

$$R_{n+1}(x) = R_n(x) - \frac{f^{(n)}(c)}{n!}(x-c)^n \qquad (\#)$$

By the induction hyp $R_n(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$

Let $u = f^{(n)}(t)$ and $dv = \frac{(x-t)^{n-1}}{(n-1)!}\,dt$.

Thus, $du = f^{(n+1)}(t)\,dt$ and $v = -\frac{(x-t)^n}{n!}$.

Hence

$$R_n(x) = -f^{(n)}(t)\frac{(x-t)^n}{n!}\Big|_c^x + \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

$$= +\frac{f^{(n)}(c)(x-c)^n}{n!} + \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

Plug this into $(\#)$ and we ~~eg~~ get

$$R_{n+1}(x) = \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt,$$

as desired. $\square$

31.6 **Corollary** Let $f: I \to \mathbb{R}$ have n derivatives on I. Let $c \in I$ and $x \in I$, $x \neq c$. $\exists y$ between x and c s.t.

$$R_n(x) = \frac{(x-c)(x-y)^{n-1} f^{(n)}(y)}{(n-1)!}$$

Pf The proof will use the IVT for Integrals (Thm 33.9 — to be covered later). It says if f is cont. on $[a,b]$ $\exists x \in (a,b)$ s.t.

$$f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx = \text{the average value of } f(x) \text{ over } [a,b].$$

Let $x < c$. (The case $x > c$ is similar.)
We apply the IVT for Integrals to the function $(x-t)^{n-1} f^{(n)}(t)$ over $[x, c]$ to get a $y \in (x,c)$ s.t.

$$(x-y)^{n-1} f^{(n)}(y) = \frac{1}{c-x}\int_x^c (x-t)^{n-1} f^{(n)}(t)\,dt.$$

Divide both sides by $(n-1)!$ and multiply by $c-x$ to get

$$\frac{(c-x)(x-y)^{n-1} f^{(n)}(y)}{(n-1)!} = \frac{\int_x^c (x-t)^{n-1} f^{(n)}(t)\,dt}{(n-1)!}$$

Thus

$$\frac{\boxed{(x-c)}(x-y)^{n-1} f^{(n)}(y)}{(n-1)!} = \frac{\int_c^x (x-t)^{n-1} f^{(n)}(t)\,dt}{(n-1)!}$$

$$= R_n(x)$$

by the last thm. $\boxed{☼}$

Now, recall the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let $a = x$ and $b = 1$. Then

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k.$$

We seek to generalize this to the case where n need not by a natural number or even an integer.

31.7 (Binomial Series Thm) Let $\alpha \in \mathbb{R}$ and $|x| < 1$. Then

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k.$$

Examples We do some examples before doing the proof.

$\alpha = 2$ $\quad (1+x)^2 = 1 + \frac{2}{1!}x + \frac{2\cdot 1}{2!}x^2 + \frac{2\cdot 1\cdot 0}{3!}x^3 + \frac{2\cdot 1\cdot 0(-1)}{4!}x^4 + \cdots$

$$= 1 + 2x + x^2.$$

$\alpha = 3$ $\quad (1+x)^3 = 1 + \frac{3}{1!}x + \frac{3\cdot 2}{2!}x^2 + \frac{3\cdot 2\cdot 1}{3!}x^3 + \frac{3\cdot 2\cdot 1\cdot 0}{4!}x^4 + \cdots$

$$= 1 + 3x + 3x^2 + x^3.$$

If $\alpha \in \mathbb{N}$ we recover the Binomial Thm.

$\alpha = -1$

$$(1+x)^{-1} = 1 + \frac{-1}{1!}x + \frac{-1(-2)}{2!}x^2 + \frac{-1(-2)(-3)}{3!}x^3 + \frac{-1(-2)(-3)(-4)}{4!}x^4 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 \cdots = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

by the Geometric Series Formula.

$\alpha = \frac{1}{2}$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{1}{2}(\frac{-1}{2})}{2!}x^2 + \frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})}{3!}x^3 + \frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{1}{2^2 2!}x^2 + \frac{3}{2^3 3!}x^3 - \frac{3 \cdot 5}{2^4 4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \sum_{k=2}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2^k k!} x^k$$

$$\sqrt{1.1} \approx 1 + \frac{.1}{2} - \frac{1}{8}(.1)^2 = 1.04875$$

Using a calculator $\sqrt{1.1} \approx 1.0488088481\ldots$

$\alpha = \frac{1}{3}$ You try this. Use the first 3 terms to estimate $(1.1)^{1/3}$.

<u>Proof</u> Let $f(x) = (1+x)^\alpha$. Then $f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))(1+x)^{\alpha-n}$.
Thus, the Taylor series of $f(x)$ centred at $c=0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!} x^k.$$

We check convergence. Let $a_k = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}$, $k = 1, 2, 3, \ldots$

If $\alpha \in \mathbb{N}$, then eventually $a_k = 0$ (k big enough) and so the series is finite and converges for all x.
Assume $\alpha \notin \mathbb{N}$. Then $a_k \neq 0$ $\forall k \in \mathbb{N}$. Thus, we can use the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\alpha - k}{k+1} \right| = 1.$$

Thus, $R = 1$ and we have convergence on $(-1,1)$.

Finally, we need to show $R_n(x) \to 0$ to be sure the Taylor series convergences to $(1+x)^\alpha$. This gets tricky.

We use Cor. 31.6: $R_n(x) = (x-c)\dfrac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y)$
where y is between c and x. In our case $c=0$.
We fix a value for $x \neq 0$. in $(-1,1)$. Then for each n there is a y_n between 0 and x s.t.

$$R_n(x) = x\frac{(x-y_n)^{n-1}}{(n-1)!} f^{(n)}(y) = \frac{x\,(x-y_n)^{n-1}}{(n-1)!} n!\, a_n (1+y_n)^{\alpha-n}$$

Thus,

$$|R_n(x)| \leq |x| \left|\frac{x-y_n}{1+y_n}\right|^{n-1} n|a_n| \, (1+y_n)^{\alpha-1}$$

Since y_n is between 0 and x, $\exists\, z \in [0,1]$ s.t. $y_n = xz$. Thus,

$$\left|\frac{x-y_n}{1+y_n}\right| = \left|\frac{x-xz}{1+xz}\right| = |x|\left|\frac{1-z}{1+xz}\right| \leq |x|$$

Since $1+xz \geq 1-z$ $\left(\Leftrightarrow xz+z \geq 0 \Leftrightarrow \cancel{z(x+1) \geq 0}\right)$
$z(x+1) \geq 0$.

Thus, $|R_n(x)| \leq |x|^n n|a_n| (1+y_n)^{\alpha-1}$.

Now, the point is to show $\lim\limits_{n\to\infty} R_n(x) = 0$.

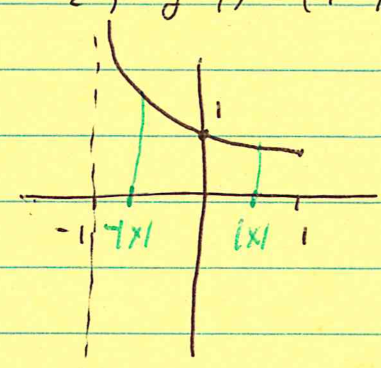
Let $s_n = n a_n x^n$ and $t_n = (1+y_n)^{\alpha-1}$.

We will show $s_n \to 0$ and that t_n is bounded.
It follows by Exercise 8.4 that $\lim\limits_{n\to\infty} s_n t_n = 0$.

t_n bdd Remember $x \in (-1,1)$ is fixed. Thus $y_n \in [-|x|, |x|] \subset (-1,1)$.

Let $g(y) = (1+y)^{\alpha-1}$. If $\alpha \geq 1$ then g is a bounded on $[-1,1]$. If $\alpha < 1$, g is unbdd as $y \to -1$, but it is bdd on $y \in [-|x|, |x|]$. That means the seq. $t_n = g(y_n)$ is bdd.

For $\alpha = \frac{1}{2}$, $g(y) = (1+y)^{-\frac{1}{2}}$



unbdd on $(-1,1)$, but bdd on $[-|x|, |x|]$.

$s_n \to 0$ We have shown $\sum a_k x^k$ converges for $|x| < 1$. Thus, $\sum k a_k x^{k-1}$ does also. Thus

$k a_k x^{k-1} \to 0$ as $k \to \infty$.

Multiplying by x does not change this. So

$s_n = n a_n x^n \to 0$ as $n \to \infty$.

$s_n t_n \to 0$ This is Exercise 8.4. Let $|t_n| < M$ $\forall n$. Let $\varepsilon > 0$. $\exists N$ s.t. $n \geq N \Rightarrow |s_n| < \frac{\varepsilon}{M}$. Then $n \geq N \Rightarrow |s_n t_n| < \varepsilon$. Thus,

$s_n t_n \to 0$.

Thus, $R_n(x) = s_n t_n \to 0$ as desired.