Ch 6 Integration

§32 The Riemann and Darboux Integrals

We are going to define two types of integrals, the Darboux (dar-boo) and the Riemann (rē-mon) integrals. It will turn out that they are equivalent. This means that the Darboux integral of $f(x)$ exists iff the Riemann integral exists and when these exist, they have the same value. The reason for doing both is that some properties of integration are easier to prove using the Darboux definition while others are easier to prove using the Riemann definition. We start with the Darboux integral.

Darboux Integral Let f be a bounded function on the closed bounded interval $[a,b]$. Let $S \subset [a,b]$. Then

- $M(f,S) = \sup\{f(x) \mid x \in S\}$
- $m(f,S) = \inf\{f(x) \mid x \in S\}$

A partition of $[a,b]$ is a finite ordered subset, $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. We write $P = \{t_0, \ldots, t_n\}$.

We define the upper and lower Darboux sums of f w.r.t. P by

- $$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$$

and

- $$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$$

Clearly $L(f,P) \le U(f,P)$.

We claim both are always real numbers (not $\mp\infty$).

Pf: $$U(f,P) \le \sum_{k=1}^{n} M(f,[a,b])(t_k - t_{k-1})$$

$$= M(f,[a,b]) \sum_{k=1}^{n} t_k - t_{k-1}$$

$$= M(f,[a,b]) \cdot (b-a) < \infty$$

Since f is bounded. Likewise,

$$L(f,P) \ge \sum_{k=1}^{n} m(f,[a,b])(t_k - t_{k-1}) = m(f,[a,b])(b-a) > -\infty$$

since f is bounded.

Thus, $$m(f,[a,b])(b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b])(b-a).$$

The <u>upper Darboux integral</u> of f over $[a,b]$ is defined to be

$$U(f) = \inf\{\, U(f,P) \mid \text{all partitions } P \text{ of } [a,b] \,\}.$$

The <u>lower Darboux integral</u> of f over $[a,b]$ is

$$L(f) = \sup\{\, L(f,P) \mid \text{all partitions } P \text{ of } [a,b] \,\}$$

Both are real numbers.

If $U(f) = L(f)$ then we say f is Darboux integrable and let $DI(f) = U(f) = L(f)$. We can also use the notation

$$DI(f) = \int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx.$$

This value is called the Darboux integral of f over $[a,b]$.

We do two standard examples. The first will use the summation formula

$$\sum_{i=1}^{n} (i)^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}.$$

Example 1 Show the Darboux integral $\int_0^1 x^2\,dx$ exists and equals $\frac{1}{3}$. Let $f(x) = x^2$.

Solution Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,1]$. Then

$$U(f,P) = \sum_{k=1}^{n} \sup\{x^2 \mid x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} t_k^2 (t_k - t_{k-1}).$$

Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$.

$$U(f) = \inf\{U(f,P) \mid \text{all partitions}\} \le \inf\{U(f,P_n) \mid n = 1,2,3,\ldots\}$$

This we can compute!

$$U(f,P_n) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{2n^3+3n^2+n}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Viewed as a sequence in n it is decreasing. Thus

$$\inf\{U(f,P_n) \mid n=1,2,3,\dots\} = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{1}{3}.$$

So, while we have not found $U(f)$ we can report that

$$U(f) \leq \frac{1}{3}.$$

A similar computation show $L(f) \geq \frac{1}{3}$.

We will prove shortly (Thm 32.4) that $L(f) \leq U(f)$ for any bounded f over $[a,b]$. Thus, we have for $f(x) = x^2$ on $[0,1]$ that

$$U(f) = L(f) = \frac{1}{3}.$$

<u>Example 2</u> Let $g(x) = \begin{cases} 1 & x \in \mathbb{Q}\cap[0,1] \\ 0 & x \notin \mathbb{Q}\cap[0,1]. \end{cases}$ Then g is a bdd function on $[0,1]$. Show that it Darboux integral does not exist.

<u>Solution</u> Let $P = \{t_0, t_1, \dots, t_n\}$ be any partition of $[0,1]$. Then

$$U(g,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1}) = \sum 1\,(t_k - t_{k-1}) = 1$$

and

$$L(g,P) = 0.$$

Thus $U(g) = 1$ and $L(f) = 0$. Hence $DI(g)$ <u>d.n.e.</u>

Let f be a bdd function on $[a,b]$.
Here are four facts about Darboux integrals we will prove.

1 <u>Lemma 32.2</u> Let $P \subseteq Q$ be partitions of $[a,b]$. Then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

2. <u>Lemma 32.3.</u> Let P and R be partitions of $[a,b]$. Then,

$$L(f,P) \le U(f,R).$$

3. <u>Thm 32.4</u> $L(f) \le U(f)$.

4. <u>Thm 32.5</u> f is Darboux integrable on $[a,b]$ iff
$\forall \varepsilon > 0$ $\exists$ a partition P of $[a,b]$ s.t.

$$U(f,P) - L(f,P) < \varepsilon.$$