

§32 continued.

Assume f is a bdd function on $[a, b]$.

Lemma 32.2 Let $P \subset Q$ be partitions of $[a, b]$. (Q is called a refinement of P .) Then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

pf

The middle inequality we already discussed.

~~We will use induction.~~ Suppose Q has one more pt than P . ~~too~~ If $P = \{t_0, t_1, \dots, t_n\}$, let

$$Q = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\}.$$

$$L(f, P) = \sum_{i=1}^n m(f, [t_{i-1}, t_i]) \cdot (t_i - t_{i-1})$$

$$L(f, Q) = \sum_{i=1}^{k-1} m(f, [t_{i-1}, t_i]) (t_i - t_{i-1}) + m(f, [t_{k-1}, u]) (u - t_{k-1}) \\ + m(f, [u, t_k]) (t_k - u) + \sum_{i=k+1}^n m(f, [t_{i-1}, t_i]) (t_i - t_{i-1})$$

$$\text{Thus, } L(f, Q) - L(f, P) = m(f, [t_{k-1}, u]) (u - t_{k-1})$$

$$+ m(f, [u, t_k]) (t_k - u) - m(f, [t_{k-1}, t_k]) (t_k - t_{k-1}) \quad (*)$$

$$\text{Now } m(f, [t_{k-1}, t_k]) \leq m(f, [t_{k-1}, u]) \text{ and} \\ m(f, [t_{k-1}, t_k]) \leq m(f, [u, t_k])$$

Thus, $m(f, [t_{k-1}, t_k]) (u - t_{k-1}) \leq m(f, [t_{k-1}, u]) (u - t_{k-1})$

and $m(f, [t_{k-1}, t_k]) (t_k - u) \leq m(f, [u, t_k]) (t_k - u)$

Add these to get

$$m(f, [t_{k-1}, t_k]) (t_k - t_{k-1}) \leq m(f, [t_{k-1}, u]) (u - t_{k-1}) + m(f, [u, t_k]) (t_k - u)$$

Thus from ~~(*)~~ we have $L(f, Q) - L(f, P) \geq 0$ or

$$L(f, P) \leq L(f, Q).$$

Now suppose Q has $m \geq 1$ more members than P .

Call them $q_1, q_2, \dots, q_m$, in any order.

Let $Q_1 = P \cup \{q_1\}$, $Q_2 = Q_1 \cup \{q_2\}$, $\dots$, $Q_m = Q_{m-1} \cup \{q_m\} = Q$.

Then $L(f, P) \leq L(f, Q_1) \leq L(f, Q_2) \leq \dots \leq L(f, Q)$.

The proof for $U(f, Q) \leq U(f, P)$ is similar. □

Lemma 323 Let P and R be partitions of $[a, b]$. Then $L(f, P) \leq U(f, R)$.

Pf Let $Q = P \cup R$. Then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, R) \quad \square$$

Thm 32.4 $L(f) \leq U(f)$.

Pf Recall $U(f) = \inf \{ U(f, Q) \mid Q \text{ is a partition of } [a, b] \}$

Let P be any partition of $[a, b]$ then since

$$L(f, P) \leq U(f, Q) \quad \forall Q, \text{ partitions of } [a, b]$$

$$L(f, P) \leq \inf \{ U(f, Q) \mid \text{all partitions } \} = U(f).$$

Recall $L(f) = \sup \{ L(f, R) \mid \text{all partitions } R \text{ of } [a, b] \}$.

For any R , $L(f, R) \leq U(f)$.

Thus $\sup \{ L(f, R) \mid \dots \} \leq U(f)$.

That is $L(f) \leq U(f)$.

Thm 32.5 (The Cauchy criterion for integrability). A bdd function f on $[a, b]$ is Darboux integrable iff

$\forall \epsilon > 0 \exists$ a partition P of $[a, b]$, s.t.

$$U(f, P) - L(f, P) < \epsilon.$$

Pf First suppose f is integrable. Let $\epsilon > 0$.

$\exists$ a partition P_1 s.t. $L(f, P_1) > L(f) - \frac{\epsilon}{2}$.

$\exists$ a partition P_2 s.t. $U(f, P_2) < U(f) + \frac{\epsilon}{2}$.

Let $P = P_1 \cup P_2$. By Lemma 32.2

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

Thus, $U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_1)$

$$< U(f) + \frac{\epsilon}{2} - \left[L(f) - \frac{\epsilon}{2} \right]$$

$$= U(f) - L(f) + \epsilon.$$

But since f is Darboux integrable, $U(f) = L(f)$.

Thus,

$$U(f, P) - L(f, P) < \epsilon.$$

Now we have to do the other direction since this is a "if and only if" theorem.

Suppose $\forall \epsilon > 0, \exists$ a partition P of $[a, b]$ s.t.

$$U(f, P) - L(f, P) < \epsilon.$$

We know that $L(f) \leq U(f)$. So, if we prove $U(f) \leq L(f)$ we know they are equal. Here we go...

$$\begin{aligned} U(f) &\leq U(f, P) = U(f, P) - L(f, P) + L(f, P) < \epsilon + L(f, P) \\ &\leq \epsilon + L(f) \end{aligned}$$

$$\text{Thus } U(f) - L(f) \leq \epsilon, \forall \epsilon > 0.$$

$$\text{Thus } U(f) - L(f) \leq 0, \text{ or } U(f) \leq L(f).$$

Hence $U(f) = L(f)$. Thus f is Darboux

integrable by definition.

