§32 cont. Riemann Integration.

Def The mesh of a partition P is the maximum length of the subintervals. That is, if $P = \{t_0, t_1, \ldots t_n\}$

$$\text{mesh}(P) = \max\{t_k - t_{k-1} \mid k = 1, 2 \ldots n\}.$$

Given a partition P a set of sample points is an ordered set $S = \{x_1, x_2, \ldots x_n\}$ s.t. $x_i \in [t_{i-1}, t_i]$. (Note, it is possible for $x_{i-1} = x_i$.) The Riemann Sum of f wrt a partition P and sample set S is

$$R(f, P, S) = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}).$$

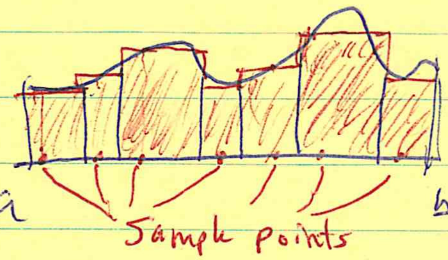
Then f is Riemann integrable over $[a,b]$ if $\exists\, r \in \mathbb{R}$ s.t.

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall$ partition P with mesh $P < \delta$ and every possible sample set

$$|R(f, P, S) - r| < \varepsilon.$$

When this happen we write $\int_a^b f = \int_a^b f(x)\,dx = r$.

Picture of a Riemann Sum



The proof that Riemann and Darboux integrability are equivalent is divided into two Theorems : 32.7 and 32.9. Then Corollary 32.10 gives a useful tool for computing Riemann integrals.

Thm 32.7 A bdd function f on $[a,b]$ is Darboux integrable iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \operatorname{mesh}(P) < \delta \Rightarrow U(f,P) - L(f,P) < \varepsilon.$$

Pf Assume this condition holds. Since we can always find a partition P with $\operatorname{mesh}(P) < \delta$, $\exists$ a partition P with $U(f,P) - L(f,P) < \varepsilon$ for any $\varepsilon > 0$. By Thm 32.5 f is Darboux integrable.

The other direction is not so easy. Suppose f is Darboux integrable on $[a,b]$.

Let $\varepsilon > 0$.

Let $P_0 = \{u_0, u_1, u_2, \dots u_m\}$ be a partition of $[a,b]$ s.t.

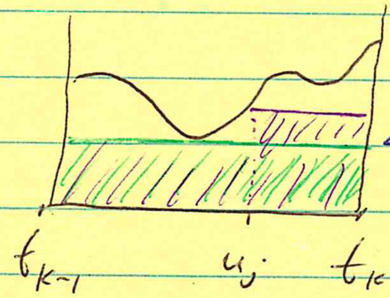
$$U(f, P_0) - L(f, P_0) < \frac{\varepsilon}{2}$$

Now, since f is bdd $\exists\ B > 0$ s.t. $|f(x)| \leq B\ \forall x \in [a,b]$.
Let $\delta = \varepsilon / 8mB$.

Let $P = \{t_0, t_1, \dots, t_n\}$ be a partition of $[a,b]$ with mesh $< \delta$.

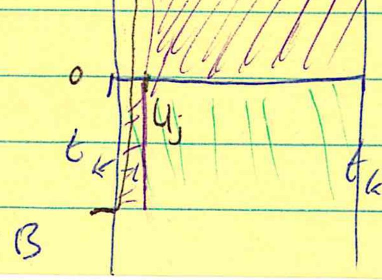
Let $Q = P \cup P_0$. We will compare $L(f,Q)$ and $L(f,P)$.

Suppose Q has just one more pt than P. So, just one subinterval has been divide into two.



$L(f,Q)$
$L(f,P)$

$L(f,Q)$ could increase. What is the most it could increase by? The interval has length at most δ. Since $|f(x)|$ is bounded by B, the "worst" that could happen is a gap of $2B$



Thus, $0 \le L(f,Q) - L(f,P) \le 2B \,\text{mesh}\, P < 2B\delta$.

If all the points in P_0 are "new", this is were not in P, then Q has m more pts than P. Thus,

$$0 \le L(f,Q) - L(f,P) \le m\,2B\delta = \frac{\varepsilon}{4}.$$

Since $P_0 \subset Q$ we know $L(f,P_0) \le L(f,Q)$.

Thus, $L(f,P_0) - L(f,P) < \frac{\varepsilon}{4}$. (it could be neg.)

You can show that likewise,

$$U(f,P) - U(f,P_0) < \frac{\varepsilon}{4}.$$

Thus, adding these two gives

$$L(f,P_0) - L(f,P) + U(f,P) - U(f,P_0) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

$$\Rightarrow \quad U(f,P) - L(f,P) < U(f,P_0) - L(f,P_0) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thm 32.9 Let f be a bdd func. on $[a,b]$. Then f is Riemann integrable iff f is Darboux integrable. When these integrals exist, they are equal.

Pf Suppose f is Darboux integrable over $[a,b]$.

Let $\varepsilon > 0$. Let $\delta > 0$ be s.t. $\text{mesh } P < \delta \Rightarrow U(f,P) - L(f,P) < \varepsilon$. $(\star)$

Let P be a partition with mesh $< \delta$. Let S be any sample set for P. We will show

$$|R(f,P,S) - D.I.(f)| < \varepsilon. \quad (☼)$$

Then, by definition, the Riemann integral exists and equals the Darboux integral.

From the definitions it is clear that

$$L(f,P) \le R(f,P,S) \le U(f,P).$$

Now, $\underbrace{U(f,P) < L(f,P) + \varepsilon}_{\text{(using }(\star))} \underbrace{\le L(f) + \varepsilon}_{\text{(property of sup)}} = D.I.(f) + \varepsilon$

and $\underbrace{L(f,P) > U(f,P) - \varepsilon}_{(\star)} \underbrace{\ge U(f) - \varepsilon}_{\text{(inf)}} = D.I.(f) - \varepsilon.$

Thus, $-\varepsilon < L(f,P) - D.I.(f) < R(f,P,S) - D.I.(f) < U(f,P) - D.I.(f) < \varepsilon.$

Therefore (☼) holds, and thus direction is proven.

Now suppose f is Riemann integrable and let $r = R.I(f)$.

We will show that $L(f) \geq r$. You will show that $U(f) \leq r$. It then follows that $L(f) = U(f) = r$ as we will have proven the theorem.

Let $\varepsilon > 0$. Let $\delta > 0$ be s.t. $\text{mesh } P < \delta \Rightarrow |R(f,P,S) - r| < \varepsilon$.

We are going to choose a special sample set S. Let $P = \{t_0, t_1, \ldots t_n\}$ be a partition of $[a,b]$ with mesh $< \delta$.

For $k = 1, 2, \ldots n$, choose $x_k \in [t_{k-1}, t_k]$ s.t.

$$f(x_k) < m(f, [t_{k-1}, t_k]) + \varepsilon.$$

(We can do this by def. of inf.)

Now,

$$R(f,P,S) \leq L(f,P) + \varepsilon(b-a).$$

Also,

$$|R(f,P,S) - r| < \varepsilon.$$

Thus,

$$L(f) \geq L(f,P) \geq R(f,P,S) - \varepsilon(b-a) > r - \varepsilon - \varepsilon(b-a) = r - \varepsilon(1 + (b-a)).$$

But, we can make ε as small as we wish.

Therefor $L(f) \geq r$ as claimed. (See the handout "Cauchy Sequences (Section 10)," top of page 7 on the ε-principle.) $\square$