

Corollary

32.10 Let f be a bdd func. on $[a, b]$. Assume $\int_a^b f$ exists.
Let (P_n) be a seq. of partitions of $[a, b]$ s.t.
$$\lim_{n \rightarrow \infty} \text{mesh } P_n = 0.$$

Then $\lim_{n \rightarrow \infty} R(f, P_n, S_n) \rightarrow \int_a^b f$, where S_n is a sample set for P_n .

pf Let $\epsilon > 0$. Let $\delta > 0$ be s.t. $|R(f, P, S) - \int_a^b f| < \epsilon$
whenever $\text{mesh } P < \delta$.

Choose N s.t. $n > N \Rightarrow \text{mesh } P_n < \delta$.

Then $|R(f, P_n, S_n) - \int_a^b f| < \epsilon$ for $n > N$.

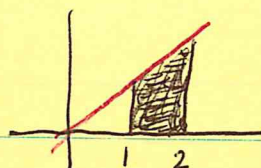
Thus, $\lim_{n \rightarrow \infty} R(f, P_n, S_n) = \int_a^b f$.

□

Note

In the next section we will show that all continuous functions on $[a, b]$ are integrable. We assume this for the next examples.

Example Compute $\int_1^2 x \, dx$.



Solution Since x is cont. this integral exists. We will use Cor. 32.10.

$$\text{Let } P_n = \{1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \dots, 1 + \frac{n}{n}\}.$$

$$\text{Let } S_n = \{1, 1 + \frac{1}{n}, \dots, 1 + \frac{n-1}{n}\} \quad (\text{the left end points})$$

$$\text{Let } R_n = R(X, P_n, S_n) = \sum_{k=0}^{n-1} (1 + \frac{k}{n}) \frac{1}{n}$$

$$= \frac{1}{n} \left(n + \frac{1}{n} \sum_{k=0}^{n-1} k \right)$$

$$= 1 + \frac{1}{n^2} \left(\frac{(n-1)(n)}{2} \right)$$

$$= 1 + \frac{n^2 - n}{2n^2}.$$

$$\text{Thus } R_n \rightarrow 1 + \frac{1}{2} = \frac{3}{2}.$$

$$\text{By Cor. 32.10 } \int_a^b x \, dx = \left(\frac{3}{2} \right).$$

Example Compute $\int_0^1 \sqrt{x} dx$.

Solution $\sqrt{x}$ is cont. so Cor 32.10 can be used.

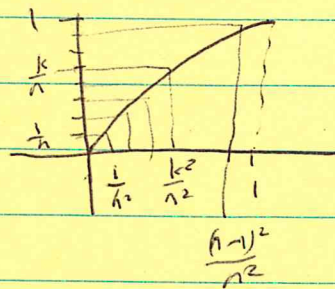
$$\text{Let } P_n = \left\{ 0, \frac{1}{n^2}, \frac{4}{n^2}, \frac{9}{n^2}, \dots, \frac{(n-1)^2}{n^2}, 1 \right\}$$

Let S_n = left end points.

$$\text{mesh } P_n = \max \left\{ \frac{k^2 - (k-1)^2}{n^2}, k=1, \dots, n \right\}$$

$$= \frac{n^2 - (n-1)^2}{n^2} = 1 - \frac{(n-1)^2}{n^2}$$

check this!



$$\lim_{n \rightarrow \infty} \text{mesh } P_n = \lim_{n \rightarrow \infty} 1 - \frac{(n-1)^2}{n^2} = 0.$$

By Cor 32.10 $R(\sqrt{x}, P_n, S_n) \rightarrow \int_0^1 \sqrt{x} dx$.

$$R(\sqrt{x}, P_n, S_n) = \sum_{k=0}^{n-1} \sqrt{\frac{k^2}{n^2}} \frac{(k+1)^2 - k^2}{n^2} = \frac{1}{n^3} \sum_{k=0}^{n-1} k(2k+1)$$

$$= \frac{1}{n^3} \left(2 \sum_{k=0}^{n-1} k^2 + \sum_{k=0}^{n-1} k \right) = \frac{1}{n^3} \left[2 \frac{(n-1)n(2(n-1)+1)}{6} + \frac{(n-1)n}{2} \right]$$

$$= \frac{4n^3 - 3n^2 - n}{6n^3} = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2} \rightarrow \frac{2}{3}.$$

$$\text{Thus, } \int_0^1 \sqrt{x} dx = \frac{2}{3}.$$