§ 33 Properties of the Riemann Integral

First we ask just which functions can we integrate? The full answer to this is beyond the scope of this course. It is covered in MATH 452 in the Riemann-Lebesgue Theorem. We will two theorems that address this issue but do not give a complete answer.

Thm 33.1 Every monotonic function f on $[a,b]$ ~~is~~ integrable.

Pf We do the increasing case. We assume f is not a constant function so $f(a) < f(b)$. f is bdd ~~by f(b)~~ below by $f(a)$ and above by $f(b)$, so it is a bdd function. We will use Thm 32.5, the "Cauchy criterion for Darboux integrability."

Let $\varepsilon > 0$. Let $P = \{t_0, \dots t_n\}$ be a partition of $[a,b]$ with

$$\text{mesh } P < \frac{\varepsilon}{f(b)-f(a)}.$$

~~Then~~ Since f is increasing on each subinterval $[t_{k-1}, t_k]$ we have

$$M(f, [t_{k-1}, t_k]) = f(t_k) \text{ and}$$

$$m(f, [t_{k-1}, t_k]) = f(t_{k-1}).$$

Thus, $U(f,P) - L(f,P) =$

$$\sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) - \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1}) \big)(t_k - t_{k-1})$$

$$\leq \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1}) \big)(\text{mesh } P)$$

$$< \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1}) \big)\left(\frac{\varepsilon}{f(b)-f(a)} \right)$$

$$= \left[\sum_{k=1}^{n} f(t_k) - f(t_{k-1}) \right] \left(\frac{\varepsilon}{f(b)-f(a)} \right)$$

$$= \big[f(t_1) - f(t_0) + f(t_2) - f(t_1) + f(t_3) - f(t_2) + \cdots + f(t_n) - f(t_{n-1}) \big]\left(\frac{\varepsilon}{f(b)-f(a)} \right)$$

$$= \big(f(\underset{\overset{\shortparallel}{b}}{t_n}) - f(\underset{\overset{\shortparallel}{a}}{t_0}) \big) \frac{\varepsilon}{f(b)-f(a)}$$

$$= \varepsilon.$$

By Thm 32.5 f is Darboux integrable and hence integrable.

$\boxtimes$

Note: Before going to the next theorem, review uniform continuity. We will be using it!

Thm 33.2 Every continuous function f on $[a,b]$ is integrable.

pf f is bdd by Thm 18.1 (pg 133). f is uniformly continuous on $[a,b]$ by Thm 19.2.

We will again use Thm 32.5. Let $\varepsilon > 0$.

Since f is uniformly cont. $\exists\ \delta > 0$ s.t.

for $x, y \in [a,b]$ with $|x-y| < \delta$ we have $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Let $P = \{t_0, t_1, \dots t_n\}$ be a partition of $[a,b]$ with mesh $P < \delta$.

By Thm 18.1, in each subinterval $[t_{k-1}, t_k]$ $\exists\ x_k, y_k$ s.t. $f(x_k) = M(f, [t_{k-1}, t_k])$ and
$f(y_k) = m(f, [t_{k-1}, t_k])$.

Since $|x_k - y_k| \le$ mesh $P < \delta$ we have

$$M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b-a}.$$

Thus,

$$U(f,P) - L(f,P) < \sum_{k=1}^{n} \left(\frac{\varepsilon}{b-a}\right)(t_k - t_{k-1}) = \left(\frac{\varepsilon}{b-a}\right) \cdot (b-a) = \varepsilon.$$

Now Thm 32.5 gives that f is integrable.

__Bizarre Example__ Let $f: [0,1] \to \mathbb{R}$ be defined by

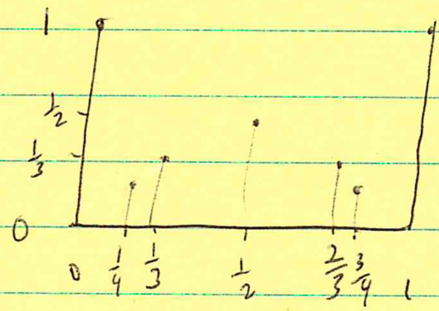
if $x \in \mathbb{Q}$, let $x = \frac{p}{q}$ be in reduced form, and $f(x) = \frac{1}{q}$,
if $x \notin \mathbb{Q}$, let $f(x) = 0$, but $f(0) = 1$.

It is called the __rational ruler function.__ We study it in MATH 452 where it is shown that

f is Riemann integrable and $\int_0^1 f(x)\,dx = 0$,

f is continuous $\forall x \in [0,1] - \mathbb{Q}$, and

f is discontinuous $\forall x \in [0,1] \cap \mathbb{Q}$.



Try to draw it!

Here are the properties of the integral:

Let f and g be integrable on $[a,b]$. Let $c \in \mathbb{R}$.
Then the following are also integrable: $cf(x)$, $f(x)+g(x)$, $f(x)g(x)$, $|f(x)|$, $\max(f(x),g(x))$, $\min(f(x),g(x))$.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx, \quad \text{if } c \in (a,b),\ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If $f(x) \le g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Some of these are theorems in the textbook and some are exercises.

<u>Thm 33.3</u> (i) $cf(x)$ is integrable and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.
(ii) $f(x)+g(x)$ is " " $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

<u>Pf</u> The book's proof uses the Darboux definition, but the proof is much easier using Riemann integration since for any partition P and sample set S

$$R(cf,P,S) = c\,R(f,P,S) \quad \text{and}$$

$$R(f+g,P,S) = R(f,P,S) + R(g,P,S).$$

The desired results are now immediate from the definition. $\blacksquare$

Thm 33.9 The Intermediate Value Theorem for Integrals.
Let f be cont. on $[a,b]$. $\exists\, c \in (a,b)$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \text{ave value of } f(x) \text{ over } [a,b].$$

Recall We used this in the proof of Corollary 31.6 (254-5).

Pf Let $M = \max\{f(x) \mid x \in [a,b]\}$, $m = \min\{f(x) \mid x \in [a,b]\}$.

If $m = M$, then $f(x)$ is a constant, $K = f(x)$. Then

$$\frac{1}{b-a}\int_a^b k\,dx = k = f(x) \text{ for any } x \in [a,b].$$

Assume $m < M$. By the Extreme Value Thm (18.1)
$\exists\, x_0, y_0 \in [a,b]$ s.t. $M = f(y_0)$ and $m = f(x_0)$.
Since $m \leq f(x) \leq M$ we have

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$f(x_0) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(y_0).$$

By the IVT (18.2) $\exists\, c$ between x_0 and y_0 s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The textbook mentions two more theorems:

The Dominated Convergence Thm (33.11) and the Monotone Convergence Thm (33.12). These are covered in MATH 452. You can skip them.