§34 FTC

FTC I

Thm 34.1 Let $g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume that g' is integrable over $[a,b]$. Then

$$\int_a^b g'(x)\,dx = g(b)-g(a).$$

Pf Since we are given that the integral exists we can apply the Cauchy criterion for Darboux integrability (Thm 32.5). Let $\varepsilon>0$. $\exists$ a partition $P=\{t_0,t_1,\ldots,t_n\}$ of $[a,b]$ s.t.

$$U(g',P)-L(g',P)<\varepsilon.$$

Next we apply the MVT to each interval $[t_{k-1},t_k]$ to show $\exists\ x_k\in(t_{k-1},t_k)$ s.t.

$$g'(x_k)=\frac{g(t_k)-g(t_{k-1})}{t_k-t_{k-1}}$$

or $$g(t_k)-g(t_{k-1}) = g'(x_k)(t_k-t_{k-1}).$$

Now,

$$g(b)-g(a)=\sum_{k=1}^{n}\left[g(t_k)-g(t_{k-1})\right] \quad \text{(telescoping series)}$$

$$=\sum_{k=n}^{n} g'(x_k)(t_k-t_{k-1})$$

Since, $m(g',[t_{k-1},t_k]) \le g'(x_k) \le M(g',[t_{k-1},t_k])$

it follows that

$$L(g', P) \leq g(b) - g(a) \leq U(g', P).$$

But we also know

$$L^*(g', P) \leq \int_a^b g'(x)\,dx \leq U^*(g', P).$$

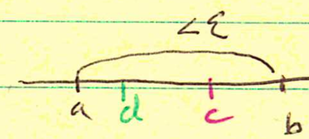
Therefore,*

$$\left| \int_a^b g'(x)\,d(x) - \big(g(b) - g(a)\big) \right| < \varepsilon.$$

Since this holds for any $\varepsilon > 0$ it must be that

$$\int_a^b g'(x)\,dx = g(b) - g(a). \quad \square$$

* Lemma: Suppose $a < c < b$, $a < d < b$ and $b - a < \varepsilon$ $(\varepsilon > 0)$. Then $|c - d| < \varepsilon$.



Pf

$$c - d < b - d$$
$$a - b < d - b$$
$$-\varepsilon < a - b < d - b < d - c$$
$$-\varepsilon < d - c$$
$$c - d < \varepsilon \quad (1).$$

$$d - c < b - c$$
$$a - b < c - b$$
$$-\varepsilon < a - b < c - b < c - d$$
$$-\varepsilon < c - d \quad (2).$$

(1) & (2) $\Rightarrow |c - d| < \varepsilon$

$\square$

Thm 34.3 (FTC II) Let f be cont. on $[a,b]$. For $x \in [a,b]$ define

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F(x)$ is cont. on $[a,b]$ and $F'(x) = f(x)$, $\forall x \in [a,b]$.

Pf Let $B > 0$ be a bound for f, that is $|f(x)| \leq B$, $\forall x \in [a,b]$

We will show that $F(x)$ is uniformly cont. on $[a,b]$. This part of the proof does not assume f is cont., merely that f be integrable on $[a,b]$.

Let $\varepsilon > 0$. For $x, y \in [a,b]$, with $x < y$ and $|x-y| < \frac{\varepsilon}{B}$ we have

$$|F(x) - F(y)| = \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right|$$

$$= \left| \int_a^x f(t)\,dt - \int_a^x f(t)\,dt - \int_x^y f(t)\,dt \right|$$

$$= \left| \int_x^y f(t)\,dt \right| \leq \int_x^y |f(t)|\,dt \leq \int_x^y B\,dt = B(y-x) < \varepsilon.$$

Thus, F is uniformly cont. on $[a,b]$.

Let $x_0 \in (a,b)$. For any $x \in [a,b]$, $x \neq x_0$, we have

$$\int_{x_0}^{x} f(t)\,dt = F(x) - F(x_0). \quad \left(\text{Recall } \int_a^b f = -\int_b^a f.\right)$$

Hence, $\dfrac{F(x)-F(x_0)}{x-x_0} = \dfrac{1}{x-x_0}\displaystyle\int_{x_0}^{x} f(t)\,dt.$

Since x_0 is fixed, $f(x_0)$ is fixed and we have

$$\int_{x_0}^{x} f(x_0)\,dt = f(x_0)\int_{x_0}^{x} 1\,dt = f(x_0)(x-x_0).$$

Thus $f(x_0) = \dfrac{1}{x-x_0}\displaystyle\int_{x_0}^{x} f(x_0)\,dt.$

Hence, $\dfrac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \dfrac{1}{x-x_0}\displaystyle\int_{x_0}^{x} f(t)\,dt - \dfrac{1}{x-x_0}\int_{x_0}^{x} f(x_0)\,dt$

$$= \frac{1}{x-x_0}\int_{x_0}^{x} f(t) - f(x_0)\,dt.$$

Now, let $\varepsilon > 0$. Since f is cont. at x_0, $\exists\, \delta > 0$ s.t.

$$t \in (a,b),\ |t-x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon.$$

Suppose $|x - x_0| < \delta$. We have two cases. Suppose $x > x_0$.
Then $\left|\dfrac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| \leq \dfrac{1}{x-x_0}\displaystyle\int_{x_0}^{x} |f(t) - f(x_0)|\,dt$

$$< \frac{1}{x-x_0}\,\varepsilon(x-x_0) \leq \varepsilon.$$

Now assume $x < x_0$. This case is slightly different.

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{1}{x - x_0} \right| \left| \int_{x_0}^{x} f(t) - f(x_0) \, dt \right|$$

$$= \frac{1}{x_0 - x} \left| \int_{x}^{x_0} f(t) - f(x_0) \, dt \right|$$

$$\leq \frac{1}{x_0 - x} \int_{x}^{x_0} |f(t) - f(x_0)| \, dt$$

$$< \frac{1}{x_0 - x} \varepsilon (x_0 - x) = \varepsilon.$$

Either way $|x - x_0| < \delta \implies$

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon.$$

By definition of a limit,

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

By definition of a derivative,

$$F'(x_0) = f(x_0).$$