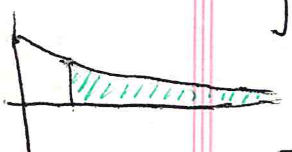
§36 Improper Integrals

Def Let $f:[a,b)\to\mathbb{R}$, $a<b\le\infty$. Suppose f is integrable on each compact interval $[a,c]$, where $a<c<b$. Then we define the improper integral of f from a to b by

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

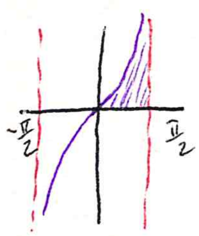
Ex



$$\int_1^\infty e^{-x}\,dx = \lim_{c\to\infty}\int_1^c e^{-x}\,dx = \lim_{c\to\infty}\left(-e^{-x}\Big|_1^c\right)$$

$$=\left(\lim_{c\to\infty} -e^{-c}\right)-\left(-e^{-1}\right) = 0+\frac{1}{e}=\frac{1}{e}$$

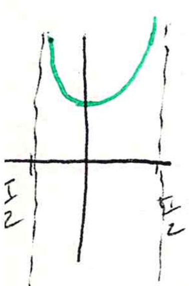
Ex



$$\int_0^{\frac{\pi}{2}} \tan x\,dx = \lim_{c\to\frac{\pi}{2}^-}\int_0^c \tan x\,dx = \lim_{c\to\frac{\pi}{2}^-} \ln(\sec x)\Big|_0^c$$

$$=\lim_{c\to\frac{\pi}{2}^-} \ln\sec(c) - \ln(\sec(0))$$

$(= \ln(1) = 0.)$



$$=\lim_{d\to\infty}\ln(d) = \infty.$$

Ex $\int_0^\infty \sin x \, dx = \lim_{c\to\infty} \int_0^c \sin x \, dx$

$= \lim_{c\to\infty} -\cos(c) + \cos(0) = -\lim_{c\to\infty} \cos(c) + 1.$

But $\lim_{c\to\infty} \cos(c)$ does not exist.

Def Let $f:(a,b] \to \mathbb{R}$ $(-\infty \le a < b)$. Suppose f is integrable on each compact interval $[c,b]$ where $a < c < b$. Then

$$\int_a^b f(x)\,dx \equiv \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

is defined to be

Ex $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \int_c^1 x^{-\frac{1}{2}}\,dx = \lim_{c\to 0^+} 2x^{\frac{1}{2}} \Big|_c^1$

$= \lim_{c\to 0^+} \left(2 - 2c^{\frac{1}{2}}\right) = 2.$

Integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ are handled by considering $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$.

If both are finite their sum is defined to be $\int_{-\infty}^{\infty} f(x)\,dx$.

If one is finite and the other is ∞, we define $\int_{-\infty}^{\infty} f(x)\,dx = \infty$.

If one is finite and the other is $-\infty$, we define $\int_{-\infty}^{\infty} f(x)\,dx = -\infty$.

If both are ∞, $\int_{-\infty}^{\infty} f(x)\,dx = \infty$.

If both are $-\infty$, $\int_{-\infty}^{\infty} f(x)\,dx = -\infty$.

If one is ∞ and the other is $-\infty$, $\int_{-\infty}^{\infty} f(x)\,dx$ is undefined (in this course). The result are not affected by the value of c.

Ex $\int_{-\infty}^{\infty} x^3\,dx$ is undefined.

If f is defined on (a,b) and is integrable on ~~any~~ every closed interval in (a,b) then

$$\int_a^b f(x)\,dx$$

is handle in a similar manner. Let $c \in (a,b)$. Then consider

$$\int_a^c f(x)\,dx \quad \text{and} \quad \int_c^b f(x)\,dx.$$

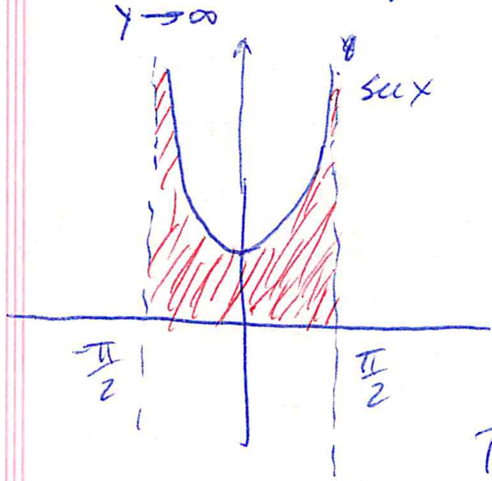
The rules are ~~jus~~ exactly analygous to what we just did.

Ex $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec x\,dx$, if it exists, is $\displaystyle\int_{-\frac{\pi}{2}}^{0} \sec x\,dx + \int_0^{\frac{\pi}{2}} \sec x\,dx$.

$$\int_0^{\frac{\pi}{2}} \sec x\,dx = \ln(\sec x + \tan x)\Big|_0^{\pi/2}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \ln(\sec x + \tan x) + \overbrace{\cancel{\ln(\underset{1}{\sec(0)} + \underset{0}{\tan(0)})}}^{=0} \qquad \boxed{\ln 1 = 0}$$

$$= \lim_{y\to\infty} \ln(y) = \infty,$$ since $\displaystyle\lim_{x\to\frac{\pi}{2}^-} \sec x = \infty$ and $\displaystyle\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$.



$\displaystyle\int_{-\frac{\pi}{2}}^{0} \sec x\,dx$ is also ∞.

Thus, $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec x\,dx = \infty$.

The basic properties of integrals carry over to improper integrals when they exist. For example

$$\int_a^\infty f(x)+g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

assuming they exist and the ~~LHS~~ RHS is not $\infty - \infty$.

Ex Here is a silly example of what can go wrong.

$$\int_0^\infty x\,dx = \infty$$

But $\displaystyle\int_0^\infty x\,dx = \int_0^\infty 2x - x\,dx = \int_0^\infty 2x\,dx - \int_0^\infty x\,dx$

$= \infty - \infty$ which is undefined.

~~Fact #8 $\int_a^\infty$ [illegible] dx~~

Ex/Fact Suppose $f(x) \geq g(x)$ on $[0,\infty)$ and $\int_0^\infty g(x)\,dx = \infty$.

Then $\int_0^\infty f(x)\,dx = \infty$ also.

The next two examples use material we have not covered, but is standard in Calc I or II courses.

Ex $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.$

Pf We first prove the integral exists and is finite.

$\displaystyle\int_{-1}^{1} e^{-x^2}\,dx$ is finite.

$$\int_{1}^{\infty} e^{-x^2}\,dx \le \int_{1}^{\infty} e^{-x}\,dx = \frac{1}{e}.$$

$$\int_{-\infty}^{-1\,\cancel{+\infty}} e^{-x^2}\,dx \le \int_{-\infty}^{\cancel{\infty}\,-1} e^{x}\,dx = \frac{1}{e}.$$

Thus, $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx$ exists.

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)$$

$$= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{+\infty}\int_{+\infty}^{+\infty} e^{-x^2} e^{-y^2}\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy = \underbrace{\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,r\,dr\,d\theta}_{\text{switch to polar}}$$

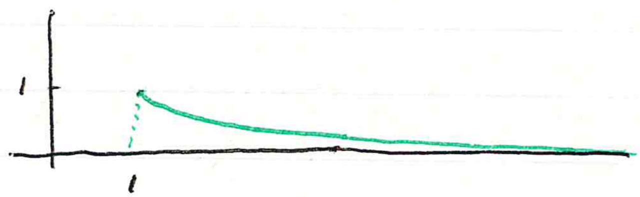
$$= 2\pi \int_{0}^{\infty} e^{-r^2}\,r\,dr = -\pi \int_{0}^{-\infty} e^{u}\,du = -\pi\left(e^{-\infty} - e^{0}\right) = \pi.$$

$u = -r^2$

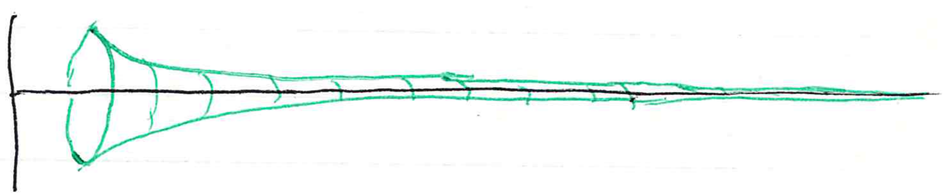
$du = -2r\,dr$

Thus, $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$ $\boxtimes$

Ex This example is called Gabriel's horn. Consider the graph of $y=\frac{1}{x}$ over $[1,\infty)$.



An example in the textbook shows $\int_1^\infty \frac{1}{x}\,dx = \infty$. Now, rotate it about the x-axis to create a surface:



This is called Gabriel's horn. We will show that its volume is π, but its surface area is infinite.

Vol. We use the disk method.

$$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \frac{-1}{x}\Big|_1^\infty = \pi\left(\frac{-1}{\infty} - \frac{-1}{1}\right) = \underline{\pi}.$$

S.A. We use the formula for surface area

$$S.A. = \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left[\left(\frac{1}{x}\right)'\right]^2}\,dx$$

$$= \int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \geq 2\pi\int_1^\infty \frac{1}{x}\,dx = \underline{\infty}.$$

$$\left(\text{since } \frac{1}{x^4} > 0\right)$$