Section 36: Improper Integrals

Example 0. I do the following pseudo-example in my calculus classes. Compute

$$\int_{-1}^{1} \frac{1}{x^2}\,dx.$$

Fake Solution.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)\Big|_{-1}^{1} = (-1) - (1) = -2.$$

Typically, the students offer no objection to this. I highlight the absurdity of this conclusion using the graph of $y = 1/x^2$. The FTC fails to apply here because the function is unbounded. Then we move on to *improper integrals.*

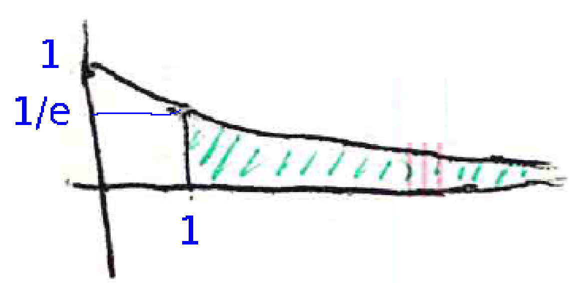
Definition. Let $f : [a,b) \to \mathbb{R}$, $a < b \leq \infty$. Suppose f is integrable on each compact interval $[a,c]$, where $a < c < b$. Then we define the *improper integral* of f from a to b as

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx,$$

when this limit exists.

Example 1.

$$\int_1^{\infty} e^{-x}\,dx = \lim_{c\to\infty} \int_1^c e^{-x}\,dx = \lim_{c\to\infty} -e^{-x}\Big|_1^c = \left(\lim_{c\to\infty} -e^{-c}\right) - (-e^{-1}) = 0 + e^{-1} = 1/e.$$

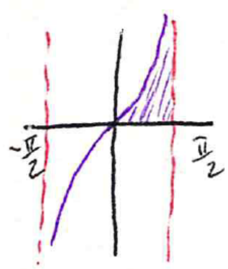


Note. In this section we will make free use of integration formulas you learned in calculus.

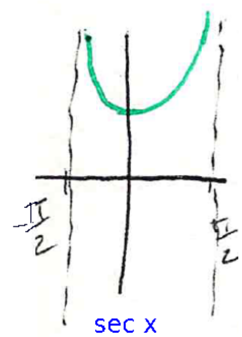
Example 2.

$$\int_0^{\pi/2} \tan x\, dx = \lim_{c \to \frac{\pi}{2}^-} \int_0^c \tan x\, dx = \lim_{c \to \frac{\pi}{2}^-} \ln \sec x \Big|_0^c$$

$$= \lim_{c \to \frac{\pi}{2}^-} \ln \sec c - \ln \sec 0 = \lim_{d \to \infty} \ln d - \ln 1 = \infty - 0 = \infty.$$



tan x



sec x

Example 3.

$$\int_0^{\infty} \sin x\, dx = \lim_{c \to \infty} \int_0^c \sin x\, dx = (\lim_{c \to \infty} -\cos c) - (-\cos 0) = -(\lim_{c \to \infty} \cos c) + 1,$$

but $\lim_{c \to \infty} \cos c$ does not exist.

Definition. Let $f : (a, b] \to \mathbb{R}$ where $-\infty \leq a < b$. Suppose f is integrable on each compact interval $[c, b]$ where $a < c < b$. Then we define

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx,$$

when this limit exist.

Example 4.

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{c \to 0^+} \int_c^1 x^{-1/2}\, dx = \lim_{c \to 0^+} 2x^{1/2} \Big|_c^1 =$$

$$\lim_{c \to 0^+} (2 - 2c^{1/2}) = 2.$$

Discussion. Integrals of the form $\int_{-\infty}^{\infty} f(x)\, dx$ for continuous functions are handled by considering $\int_{-\infty}^{c} f(x)\, dx$ and $\int_c^{\infty} f(x)\, dx$. If both are finite their sum is defined to be $\int_{-\infty}^{\infty} f(x)\,dx$. If one is finite and the other is infinity we set $\int_{-\infty}^{\infty} f(x)\,dx = \infty$. If one is finite and the other is $-\infty$ we set $\int_{-\infty}^{\infty} f(x)\,dx = -\infty$. If both are ∞ we set $\int_{-\infty}^{\infty} f(x)\,dx = \infty$. If both are $-\infty$ we set $\int_{-\infty}^{\infty} f(x)\,dx = -\infty$. If one is ∞ and the other is $-\infty$ than $\int_{-\infty}^{\infty} f(x)\,dx$ is left undefined. These results are not effected by the value of c.

Example 5. $\int_{-\infty}^{\infty} x^3\,dx$ is undefined.

Example 6. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec x\,dx$, if it exists, is

$$\int_{-\frac{\pi}{2}}^{0} \sec x\,dx + \int_{0}^{\frac{\pi}{2}} \sec x\,dx.$$

These are both infinity.

$$\int_{0}^{\frac{\pi}{2}} \sec x\,dx = \lim_{c \to \frac{\pi}{2}^-} \ln(\sec x + \tan x)\Big|_0^c = \lim_{c \to \frac{\pi}{2}^-} \ln(\sec x + \tan x) - \ln(\sec 0 + \tan 0)$$

$$= \lim_{d \to \infty} \ln d + \ln 1 = \infty.$$

You can check $\int_{-\frac{\pi}{2}}^{0} \sec x\,dx = \infty$. Thus,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec x\,dx = \infty.$$

Basic properties of integrals carry over to improper integrals. For example

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

when the integrals exists and the righthand side is not $\infty - \infty$.

Example 7. Here is a silly example of what can go wrong. $\int_0^\infty x\,dx = \infty$, but

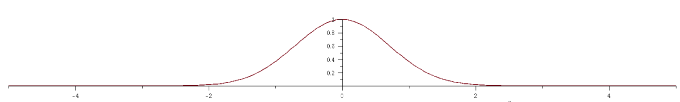
$$\int_0^\infty x\,dx = \int_0^\infty 2x - x\,dx = \int_0^\infty 2x dx - \int_0^\infty x\,dx = \text{“}\infty - \infty\text{”}$$

which is undefined.

It is easy to show that if $f(x) \geq g(x)$ on (a, b) and $\int_a^b g(x)\,dx = \infty$ then $\int_a^b f(x)\,dx = \infty$. See textbook. Note, since $\sec x \geq \tan x$ on $[0, \pi/2)$, Example 6 can be done quickly using Example 2.

The next two examples use material we have not covered, but is standard in Calc II or Calc III courses.

Example 8. $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$. See graph below. It called a *Gaussian function* or less formally a *bell curve* and is used in statistics.



Proof. We first prove that the integral exists an is finite.

$$\int_{-1}^1 e^{-x^2}\,dx,$$

clearly exists and is finite.

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x} = 1/e.$$

$$\int_{-\infty}^{-1} e^{-x^2}\,dx \leq \int_{-\infty}^{-1} e^{x} = 1/e.$$

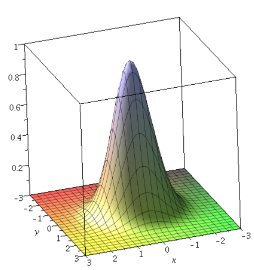
Thus $\int_{-\infty}^\infty e^{-x^2}\,dx$ exists and is finite. Now,

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta \qquad \text{(convert to polar)} \\
&= 2\pi\int_0^{\infty} e^{-r^2} r\,dr \\
&= -\pi\int_0^{-\infty} e^{u}\,du \qquad \text{(u-substitution)} \\
&= -\pi(e^{-\infty} - e^{0}) = \pi
\end{aligned}$$

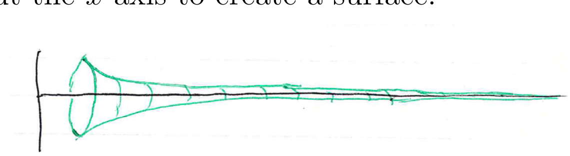
Thus,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Note. Along the way we have shown that the volume under the graph of the 2-dimensional Gaussian $z = e^{-r^2}$ is π.



Example 9. Consider the graph of $y = 1/x$ over $[1, \infty)$. Now, rotate it about the x-axis to create a surface.



This is called *Gabriel's horn.* We show that its volume is π, but its surface area is infinite.

Volume. We use the disk method.

$$V = \int_1^\infty \pi \left(\tfrac{1}{x}\right)^2 \, dx = \lim_{c\to\infty} \int_1^c \pi \left(\tfrac{1}{x}\right)^2 \, dx = \lim_{c\to\infty} \frac{-\pi}{x}\bigg|_1^c = \pi.$$

Surface Area. We use the formula for the surface area of a revolution.

$$\begin{aligned} S &= \int_1^\infty \tfrac{2\pi}{x} \sqrt{1 + \left[\left(\tfrac{1}{x}\right)'\right]^2} \, dx \\ &= \int_1^\infty \tfrac{2\pi}{x} \sqrt{1 + \tfrac{1}{x^4}} \, dx \geq 2\pi \int_1^\infty \tfrac{1}{x} \, dx = \infty. \end{aligned}$$

The joke is, if you want to paint Gabriel's horn, you buy a π units of paint and pour them in.