The Weierstrass M-Test
Section 25

The main result from Section 25 is the Weierstrass M-Test. There is some other material on integration that we will come back to later (after Test 2).

The proof of the Weierstrass M-Test uses Exercise 8.9. I'll do that first.

Theorem (Exercise 8.9). Suppose $s_n \to s$ and that $\exists N$ s.t. $n > N$ implies $s_n \in (a, b)$. Then, $s \in [a, b]$.

Proof. Suppose $s_n \to s \notin [a, b]$.

If $s < a$, let $\epsilon = (a - s)/2$.

If $s > b$, let $\epsilon = (s - b)/2$.

Either way, $\exists\, N$ s.t. $n > N$ implies $s_n \notin [a, b]$. □

Definition. Let (f_n) be a sequence of real valued functions defined on a set $S \subset \mathbb{R}$. Then we say (f_n) is **uniformly Cauchy** if

$$\forall\, \epsilon > 0\, \exists\, N \text{ s.t. } n \geq m > N \,\&\, x \in S \implies |f_n(x) - f_m(x)| < \epsilon.$$

Theorem. Let (f_n) be uniformly Cauchy on $S \subset \mathbb{R}$. The $\exists\, f : S \to \mathbb{R}$ s.t. (f_n) converges uniformly to f on S.

Proof. For each $x \in S$ the sequence $(f_n(x))$ is Cauchy, and thus has a limit. Define a function $f : S \to \mathbb{R}$ by $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

Let $\epsilon > 0$. Then $\exists\, N$ s.t.

$$m, n > N \,\&\, x \in S \implies |f_n(x) - f_m(x)| < 0.9\epsilon.$$

For now fix an $x \in S$ and an $m > N$. Then $n > N$ implies

$$f_n(x) \in (f_m(x) - 0.9\epsilon, f_m(x) + 0.9\epsilon).$$

Thus, by Exercise 8.9 we have

$$f(x) \in [f_m(x) - 0.9\epsilon, f_m(x) + 0.9\epsilon].$$

Thus,

$$|f(x) - f_m(x)| \leq 0.9\epsilon < \epsilon, \;\; \forall x \in S,\, m > N.$$

Thus, $f_n \rightrightarrows f$ on S. □

Theorem (Weierstrass M-Test). Let (M_k) be a sequence of nonnegative real numbers such that $\sum_{k=1}^{\infty} M_k < \infty$. Let (g_k) be a sequence of real valued functions on $S \subset \mathbb{R}$.

If $|g_k(x)| \leq M_k \;\forall\, x \in S$, then $\sum g_k$ converges uniformly on S.

Proof. Let $\epsilon > 0$. Then $\exists\, N$ s.t. $n \geq m > N$ implies $\sum_{k=m}^{n} M_k < \epsilon$.

Let $f_n(x) = \sum_{k=1}^{n} g_k(x)$. Then

$$|f_n(x) - f_{m-1}(x)| = \left| \sum_{k=m}^{n} g_k(x) \right| \leq \sum_{k=m}^{n} |g_k(x)| \leq \sum_{k=m}^{n} M_k < \epsilon.$$

Thus, the series converges uniformly on S. □

The main application we will see is to prove that limits of power series are continuous.