# Algebraic and Transcendental Numbers

**Definitions.** A real (or complex) number is **algebraic** if it is a root of a polynomial with integer coefficients. A real (or complex) number that is not algebraic is a **transcendental**. We will only be working with real numbers.

**Examples.** All rational numbers are algebraic. The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt{2+\sqrt[3]{3}}$ are algebraic.

Are there any transcendental numbers? Yes, $e$ and $\pi$ are transcendental. We will not prove this now.[1]

**Theorem.** The set of algebraic numbers is countable. It follows that the set of transcendental numbers is uncountable.

*Proof.* Let $\mathcal{A}$ be the set of algebraic numbers. Let $P$ be the set of polynomials with integer coefficients. We will first show that $P$ is countable.

Let $P_n$ = polynomials of degree $n$ or less with integer coefficients. The map $P_n \to \mathbb{Z}^{n+1}$, given by

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mapsto (a_n, a_{n-1}, \ldots, a_1, a_0),$$

is a bijection. Since $\mathbb{Z}^{n+1}$ is countable so is $P_n$.

Now, since $P = \cup_{n=1}^{\infty} P_n$, we know that $P$ is countable.

For every $a \in \mathcal{A}$ there exists a $p \in P$ such that $p(a) = 0$. Let $c : \mathcal{A} \to P$ assign to each $a \in \mathcal{A}$ a $p \in P$ such that $p(a) = 0$. (Notice that $c$ is a choice function.) Unfortunately, $c$ is not one-to-one. But, we can get around this. Let $R = c(\mathcal{A}) \subset P$, that is $R$ is the range of $c$. For each $p \in R$ we know that $c^{-1}(p)$ is a finite set. (Why?) Now

$$\mathcal{A} = \bigcup_{p \in R} c^{-1}(p)$$

is a countable union of finite sets and hence is countable. $\square$

Note: The countability of $\mathcal{A}$ can be established without invoking the Axiom of Choice, but the proof is a bit more cumbersome.

[1] Proofs that $e$ and $\pi$ are irrational are given on pages 117 and 118, respectively, in our textbook, *Elementary Analysis* by Ross. A proof that $e$ is transcendental is in *Topics in Algebra* by Herstein. It is elementary but convoluted. A proof that $\pi$ is transcendental is in *Algebra* by Lang. It involves rather advanced concepts.