Arc Length

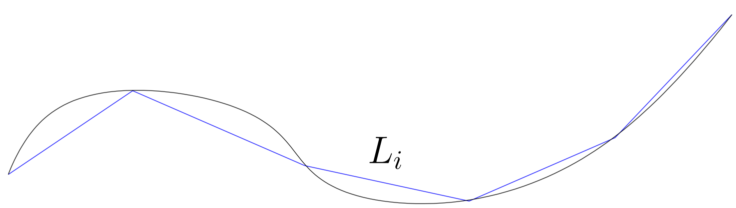
In Part I we establish a formula for computing the arc length of the graph of function

$$f : [a, b] \to \mathbb{R}.$$

In Part II we establish a formula for computing the arc length of the graph of parametric function

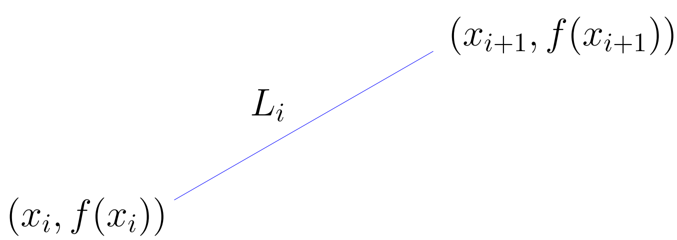
$$\mathbf{r} : [a, b] \to \mathbb{R}^2.$$

Part I. Let $f : [a, b] \to \mathbb{R}$. We will establish a formula for the length of the graph of $y = f(x)$ over $x \in [a, b]$. We will assume $f'(x)$ exists.



Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$.

Let $L_i = \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}$.



The sum $\sum_{i=0}^{n-1} L_i$ can be regarded as an approximation to the length of the graph.

But, this is not in the form of a Riemann sum. We will rectify this using the MVT.

For each $i = 0, \ldots, n-1$, $\exists\, t_i \in (x_i, x_{i+1})$ s.t.

$$f'(t_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}.$$

Thus, $f(x_{i+1}) - f(x_i) = f'(t_i)(x_{i+1} - x_i)$.

Now,

$$L_i = \sqrt{(x_{i+1} - x_i)^2 + [f'(t_i)(x_{i+1} - x_i)]^2} = \sqrt{1 + [f'(t_i)]^2}\,(x_{i+1} - x_i).$$

Thus, $\sum_{i=0}^{n-1} L_i$ is now in the form of a Riemann sum.

If $\sqrt{1 + [f'(x)]^2}$ is Riemann integrable over $[a, b]$, then

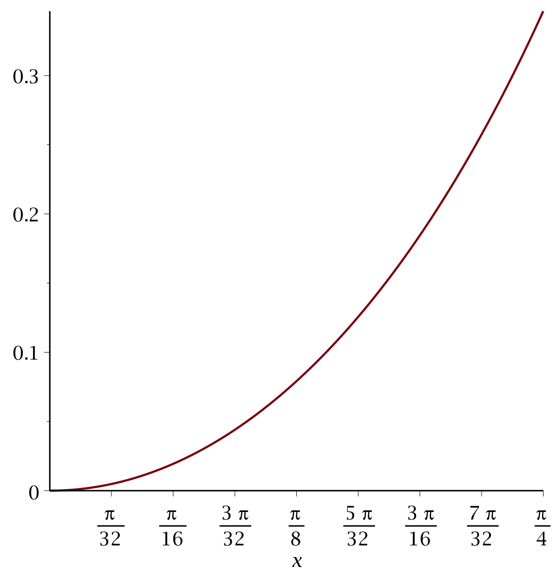
$$\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

exists and is called the **arc length** of the graph of $y = f(x)$ over $a \leq x \leq b$.

Example. Find the arc length of the graph of $y = \ln \sec x$ over $0 \leq x \leq \pi/4$.

Solution.

$$
\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1+(y')^2}\,dx \\
&= \int_0^{\pi/4} \sqrt{1+\left(\frac{\sec x \tan x}{\sec x}\right)^2}\,dx \\
&= \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx \\
&= \int_0^{\pi/4} \sec x\,dx \\
&= \ln|\sec x + \tan x|\Big|_0^{\pi/4} \\
&= \ln\left(\sqrt{2}+1\right) - \ln(1-0) = \ln\left(\sqrt{2}+1\right) \approx 0.88
\end{aligned}
$$



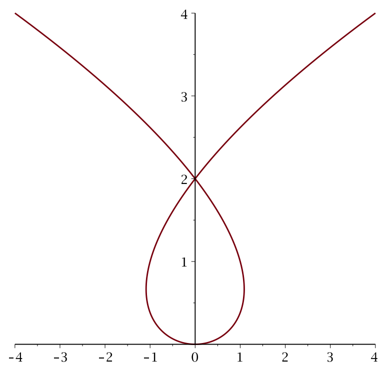
Part II. Next we consider the more challenging problem of finding the arc length of a curve in $\mathbb{R}^2$ given parametrically.

$$\mathbf{r} : [a,b] \to \mathbb{R}^2,$$

$$\mathbf{r}(t) = (x(t), y(t)).$$

We will assume $x(t)$ and $y(t)$ have continuous first derivatives.

Example. The graph of $\mathbf{r}(t) = (t^3 - 2t, t^2)$ for $-2 \le t \le 2$ is shown below.



Note. We will really be finding the "distance traveled," since the path traveled may cover parts of the curve more than once. For example, let $\mathbf{r}(t) = (t(t-1)^2, t(t-1)^2)$ for $0 \le t \le 2$. The curve is just the line segment from (0,0) to (2,2). But, the path traveled passes through the lower part of this segment 3 times. Convince yourself of this.

Now, let $\{t_0, t_1, \dots, t_n\}$ be a partition on $[a, b]$. Let

$$L_i = \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2}.$$

Then $\displaystyle\sum_{i=0}^{n-1} L_i$ should approximate the desired length. Again, it is not in the form of a Riemann sum.

By the MVT $\exists$ values t_i^* and t_i^{**} in each (t_i, t_{i+1}) s.t.

$$x'(t_i^*)(t_{i+1} - t_i) = x(t_{i+1}) - x(t_i),$$

$$y'(t_i^{**})(t_{i+1} - t_i) = y(t_{i+1}) - y(t_i).$$

Thus,

$$L_i = \sqrt{[x'(t_i^*)]^2 + [y'(t_i^{**})]^2}\,(t_{i+1} - t_i).$$

Since t_i^* need not be equal to t_i^{**}, we still do not have that $\displaystyle\sum_{i=0}^{n-1} L_i$ is in the form of a Riemann sum. However, we claim this sum does converge (in the sense of Riemann integration) to

$$\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

Let $\epsilon > 0$. Let $I = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$.

We will show $\exists\ \delta > 0$ s.t.

$$\left| I - \sum_{i=0}^{n-1} L_i \right| < \epsilon$$

for any partition with mesh $< \delta$ and any pair of test point sets, $\{t_0^*, t_1^*, \ldots, t_n^*\}$ and $\{t_0^{**}, t_1^{**}, \ldots, t_n^{**}\}$.

Let $P_i = \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\,(t_{i+1} - t_i)$.

$\exists\ \delta_1 > 0$ s.t. $\left| I - \sum_{i=0}^{n-1} P_i \right| < \epsilon/2$ for any partition with mesh $< \delta_1$ and any set of test points $\{t_0^*, t_1^*, \ldots, t_n^*\}$.

$\exists\ \delta_2 > 0$ s.t. $|y'(t_i^*) - y'(t_i^{**})| < \frac{\epsilon}{2(b-a)}$, whenever $|t_i^* - t_i^{**}| < \delta_2$ since $y'(t)$ is uniformly continuous over $[a, b]$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Now we can report that

$$\begin{aligned} \left| I - \sum_{i=0}^{n-1} L_i \right| &= \left| I - \sum_{i=0}^{n-1} P_i + \sum_{i=0}^{n-1} P_i - \sum_{i=0}^{n-1} L_i \right| \\ &\leq \left| I - \sum_{i=0}^{n-1} P_i \right| + \left| \sum_{i=0}^{n-1} P_i - \sum_{i=0}^{n-1} L_i \right| \\ &\leq \epsilon/2 + \sum_{i=0}^{n-1} \left| P_i - L_i \right| . \end{aligned}$$

We pause to study $|P_i - L_i|$.

$$|P_i - L_i| = \left| \sqrt{[x'(t_i^*)]^2 - [y'(t_i^*)]^2} - \sqrt{[x'(t_i^*)]^2 - [y'(t_i^{**})]^2} \right| (t_{i+1} - t_i).$$

From Exercise 1 we have

$$\left| \sqrt{a^2 + b^2} - \sqrt{a^2 + c^2} \right| \leq |b - c|.$$

Thus,

$$|P_i - L_i| \leq |y'(t_i^*) - y'(t_i^{**})|(t_{i+1} - t_i) < \frac{\epsilon}{2(b-a)}(t_{i+1} - t_i).$$

Now we have

$$\sum_{i=0}^{n-1} |P_i - L_i| \leq \frac{\epsilon}{2(b-a)} \sum_{i=0}^{n-1} (t_{i+1} - t_i) = \epsilon/2.$$

Therefore,

$$\left| I - \sum_{i=0}^{n-1} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This justifies defining the arc length of a smooth parametric path by

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Exercises.

1. Prove that $\left|\sqrt{a^2+b^2} - \sqrt{a^2+c^2}\right| \leq |b-c|$.

2. Show that the formula, $L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$, for the arc length of a graph can be derived from the formula for the arc length of a parametric path.

3. Use a computer integration program to find the arc length of $\mathbf{r}(t) = (t^3 - 2t, t^2)$ for $-2 \leq t \leq 2$.

References.

Introduction to Calculus and Analysis, Volume One, by Richard Courant and Fritz John, Wiley, 1965.