Completeness Axiom

Definitions. Let S be a nonempty subset of $\mathbb{R}$. An **upper bound** for S is any real number b such that if $x \in S$ then $x \leq b$. A **least upper bound (lub)** for S is any real number c such that c is an upper bound for S and if b is another upper bound of S then $c < b$. If S contains its least upper bound c then we say c is the **maximum** of S.

Examples. Let $S = (-1, 1)$. Then 17 is an upper bound of S and 1 is the least upper bound of S. In this case S does not have a maximum. Let $T = (0, 1) \cup \{2\}$. Then 2 is the least upper bound and the maximum of T.

The terms **lower bound**, **greatest lower bound (glb)** and **minimum** are defined similarly.

Examples. Let $A = (0, 3) \cup \mathbb{N}$. Then A does not have an upper bound. Any negative number is a lower bound and 0 is the greatest lower bound. The set A does not have a minimum.

Let $B = \left\{\frac{n+1}{n} \,|\, n \in \mathbb{N}\right\}$. Then 2 is the maximum and 1 is the greatest lower bound. The set B does not have a minimum.

The Completeness Axiom for $\mathbb{R}$. If S is a nonempty subset of $\mathbb{R}$ that is bounded from above then S has a least upper bound.

In MATH 352 the Completeness Axiom is assumed to be true for $\mathbb{R}$. In MATH 452 we prove the Completeness Axiom is true for $\mathbb{R}$.

The Completeness Axiom is **false** for $\mathbb{Q}$. Let $A = \{r \in \mathbb{Q} \,|\, r^2 < 2\}$. Then A is not empty since $1 \in A$. You can show that 3 is an upper bound for A. But A does not have a least upper bound in $\mathbb{Q}$. If it did, say r_* is the lub, then it can be shown that $(r_*)^2 = 2$, but we know there is no such rational number. If we regard A as a subset of $\mathbb{R}$ then it does have a lub that is called $\sqrt{2}$. In MATH 452 we prove that $(\sqrt{2})^2 = 2$. In MATH 352 we assume this.

Corollary. Using the Completeness Axiom it is easy to prove that if S is a nonempty subset of $\mathbb{R}$ that is bounded from below then S has a greatest lower bound. See textbook for proof.

Definitions. Let $S \subset \mathbb{R}$. Then the **supremum** and **infimum** of S are defined as follows.

$$\sup S = \begin{cases} +\infty & \text{if } S \neq \emptyset \text{ and is not bounded above,} \\ \text{lub } S & \text{if } S \neq \emptyset \text{ and is bounded above,} \\ -\infty & \text{if } S = \emptyset. \end{cases}$$

$$\inf S = \begin{cases} -\infty & \text{if } S \neq \emptyset \text{ and is not bounded below,} \\ \text{glb } S & \text{if } S \neq \emptyset \text{ and is bounded below,} \\ +\infty & \text{if } S = \emptyset. \end{cases}$$

Theorem. The Archimedean Property. Let a and b be positive real numbers. Then $\exists\ n \in \mathbb{N}$ such that $na > b$.

The proof uses the Completeness Axiom and is harder than you would think!

Proof. Suppose not. Then $\exists\ a > 0$ and $b > 0$ such that $\forall\ n \in \mathbb{N}$, $na \leq b$. Let

$$S = \{na \,|\, n \in \mathbb{N}\}.$$

Since b is an upper bound for S, S must have a lub. Call it s_0.

Since $0 < a$ we have $s_0 < s_0 + a$ and hence $s_0 - a < s_0$.

$\exists\ n_0 \in \mathbb{N}$ such that $s_0 - a < n_o a$ because $s_0 - a$ is less than the least upper bound of S.

Hence, $s_0 < (n_0 + 1)a$.

But this means $(n_0 + 1)a \notin S$. Hence, our supposition was foolish! The Archimedean Property has been vindicated!! $\square$

Two Corollaries. These will be useful in proofs.
1. If $a > 0$, then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < a$.
2. If $b > 0$, then $\exists\, n \in \mathbb{N}$ s.t. $n > b$.

Proofs.
1. Let $a > 0$ and $b = 1.0$. By the Archimedean Property $\exists\, n \in \mathbb{N}$ s.t. $na > 1$. Then $\frac{1}{n} < a$.
2. Let $a = 1 > 0$ and $b > 0$. By the Archimedean Property $\exists\, n \in \mathbb{N}$ s.t. $n \cdot 1 > b$. Then $n > b$.

Denseness of $\mathbb{Q}$ in $\mathbb{R}$. $\forall\, a$ and b in $\mathbb{R}$, with $a < b$, $\exists\, \frac{m}{n} \in \mathbb{Q}$ s.t.

$$a < \tfrac{m}{n} < b.$$

Proof. Since $b - a > 0$, $\exists\, n \in \mathbb{N}$ s.t. $n(b-a) > 1$ (*).

$\exists\, k \in \mathbb{N}$ s.t. $-k < na < nb < k$. (Why?)

Let m be the smallest number in $\{-k, -k+1, \ldots, k-1, k\}$ that is bigger than na. Then

$$-k < na < m \quad \text{and} \quad m - 1 \leq na.$$

Thus, using (*),

$$na < m \leq na + 1 < nb.$$

Thus,

$$a < \tfrac{m}{n} < b.$$

□