Finite and Infinite Sets

This material is not in the Ross textbook. It can be found in Chapter 5 of *A Transition to Advance Mathematics*, by Smith, Eggen & Andre, the textbook that is often used in MATH 302.

Definitions. A set is **finite** if it is empty or can be put into one-to-one correspondence with a set of the form $\{1, 2, 3, \ldots, n\}$. A set is **countably infinite** if it can be put into one-to-one correspondence with $\mathbb{N}$. A set is **countable** is it is finite or countably infinite. A set is **uncountable** if it is not finite and cannot be put into one-to-one correspondence with $\mathbb{N}$.

Facts. You should know from MATH 302 how to show that $\mathbb{Q}$ is countably infinite and that $\mathbb{R}$ is uncountable.

Mystery. Is there a set whose cardinality is strictly in between the cardinalities of $\mathbb{N}$ and $\mathbb{R}$? Lookup the Continuum Hypothesis.

Theorem 1. A finite union of finite sets is finite.

Theorem 2. A countable union of finite sets is countable.

Theorem 3. A finite union of countable sets is countable.

Theorem 4. A countable union of countable sets is countable.

Theorem 5. A finite product of finite sets is finite.

Theorem 6. A finite product of countable sets is countable.

Theorem 7. But, countably infinite products of nonempty finite sets are not countable.

I will assume you have seen in MATH 302 Theorems 1, 2, 3, and 5. Here is an example illustrating Theorem 7. Let

$$X = \prod_{i=1}^{\infty} \{0, 1\} = \{(s_1, s_2, s_3, \ldots) \mid s_i = 0 \text{ or } 1\}.$$

We define an onto map $f : X \to [0, 1]$ by

$$f(s_1, s_2, s_3, \dots) = 0.s_1 s_2 s_3 \dots,$$

where the right side is in base two. Since $[0, 1]$ is uncountable so is X. (This assumes every number in [0,1] has a base two expansion.)

We will next prove Theorem 6 and then use it to prove Theorem 4.

Proof of Theorem 6. Let $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$. We will show that $A \times B$ is countable.

Let $C_n = \{(a_i, b_j) \in A \times B \mid i + j = n\}$. Each C_n is finite. Since

$$A \times B = \bigcup_{n=2}^{\infty} C_n$$

Theorem 2 shows that $A \times B$ is countable. If A or B was finite the same argument works.

Let A_i, for $i \in \mathbb{N}$, be countable. We know now that $A_1 \times A_2$ is countable. Suppose for some $k > 1$ we have that $A_1 \times \cdots \times A_k$ is countable. Since $A_1 \times \cdots A_k \times A_{k+1}$ is equivalent to $(A_1 \times \cdots A_k) \times A_{k+1}$ it is also countable. □

Proof of Theroem 4. Let A_i, for $i \in \mathbb{N}$, be countably infinite. Assume for now that they are disjoint. Let A be their union. For each $i \in \mathbb{N}$ let

$$f_i : \mathbb{N} \to A_i$$

be a bijection. Define

$$h : \mathbb{N} \times \mathbb{N} \to A \quad \text{by} \quad h(m, n) = f_m(n).$$

We claim h is a bijection.

We first show that h is onto. Let $x \in A$. Then $\exists\, m \in \mathbb{N}$ such that $x \in A_m$. Also, $\exists\, n \in \mathbb{N}$ such that $f_m(n) = x$, since f_m is onto. Thus, $h(m, n) = x$.

Next we show that h is one-to-one. Suppose $h(m, n) = h(p, q)$. If $m \neq p$, then $f_m(n) \neq f_p(q)$, since $A_m \cap A_p = \emptyset$. Suppose $m = p$, but $n \neq q$. Then $f_m(n) \neq f_m(q)$, since f_m is one-to-one. Thus, $f_m(n) \neq f_p(q)$. Thus, h is one-to-one and hence a bijection as claimed.

By Theorem 6 we know $\mathbb{N} \times \mathbb{N}$ is countable, hence A is.

Finally, we drop the assumption that the A_i are disjoint. Let $A'_1 = A_1$ and for $i \geq 2$ let $A'_i = A_i - \cup_{j=1}^{i-1} A_j$. Now the A'_i are disjoint and

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i.$$

Some of the A'_i could be finite. If an A'_i is empty it can be skipped and the remaining ones renumbered. If an A'_i is finite the map f_i will now be of the form $f_i : \{1, 2, \ldots, n\} \to A'_i$. Then h will only be defined on a subset of $\mathbb{N} \times \mathbb{N}$, but the same argument still goes through. □

The proof of Theorem 4 used, in a subtle way, something called the **Axiom of Choice**. When mathematicians were first trying to formalize the ideas of set theory and logic they ran into various paradoxes. To avoid these they eventually established an axiomatic structure for set theory. Out of several approaches the Zermelo-Fraenkel axioms became the most popular. The following question arose. Given a collection of nonempty sets is there a function that selects just one member of each set? If the collection is finite one can use the Zermelo-Fraenkel axioms to prove such a function, called a **choice function**, exists. But, if the collection is infinite no one was able to show that a choice function exists. The assumption that choice functions do exist is called the Axiom of Choice.

It is now known that the Axiom of Choice cannot be derived from the Zermelo-Fraenkel axioms and that it cannot be contradicted by them. It is taken as an independent axiom.

In the proof of Theorem 4 we said there exists for each i a bijection $f_i : \mathbb{N} \to A_i$. But there are many such functions. For each i we had to choose one to construct h. Hence we employed the Axiom of Choice.

Lookup the Axiom of Choice and read about the history of it.