# Ordered Fields

**Definition.** Let $S$ be a set. A **binary operation** is a map from $S \times S$ to $S$.

**Definition.** An **ordered field** is a set $\mathbb{F}$ with more than one member together with two binary operations, addition $+$ and multiplication $\cdot$ and an order relation $\leq$ satisfying the axioms below for all $a$, $b$, and $c$ in $\mathbb{F}$. (The $\cdot$ is often left unwritten.)

**A1.** $a + (b + c) = (a + b) + c$.
**A2.** $a + b = b + a$.
**A3.** $\exists 0 \in \mathbb{F}$ such that $a + 0 = a$.
**A4.** $\exists -a \in \mathbb{F}$ such that $a + (-a) = 0$.
**M1.** $a(bc) = (ab)c$.
**M2.** $ab = ba$.
**M3.** $\exists 1 \in \mathbb{F}$ such that $a \cdot 1 = a$.
**M4.** If $a \neq 0$, $\exists a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$.
**DL.** $a(b + c) = ab + ac$.
**O1.** Either $a \leq b$ or $b \leq a$ is true.
**O2.** If $a \leq b$ and $b \leq a$, then $a = b$.
**O3.** If $a \leq b$ and $b \leq c$, then $a \leq c$.
**O4.** If $a \leq b$, then $a + c \leq b + c$.
**O5.** If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Note: Given $\leq$ then $\geq$, $<$ and $>$ are defined as usual.

From these beginnings the following can be proven.

**Theorem 3.1** Let $\mathbb{F}$ be a field. Then $\forall$ $a$, $b$, $c$ in $\mathbb{F}$ the following hold.

**(i)** If $a + c = b + c$, then $a = b$.
**(ii)** $a \cdot 0 = 0$.
**(iii)** $(-a)b = -(ab)$.
**(iv)** $(-a)(-b) = ab$.
**(v)** If $ac = bc$ and $c \neq 0$, then $a = b$.
**(vi)** If $ab = 0$ then either $a = 0$ or $b = 0$.

**Theorem 3.2** Let $\mathbb{F}$ be an ordered field. Then $\forall$ $a$, $b$, $c$ in $\mathbb{F}$ the following hold.

**(i)** If $a \leq b$, then $-b \leq -a$.
**(ii)** If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.
**(iii)** If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.
**(iv)** $0 \leq a^2$.
**(v)** $0 < 1$.
**(vi)** If $0 < a$, then $0 < a^{-1}$.
**(vii)** If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

**Some Proofs.**
**3.1(i)** If $a + c = b + c$, then $a = b$.

$$\begin{array}{rcll} a+c &=& b+c & \text{is given.} \\ (a+c)+(-c) &=& (b+c)+(-c) & \text{by def. of binary op.} \\ a+(c+(-c)) &=& b+(c+(-c)) & \text{by axiom A1 applied to both sides.} \\ a+0 &=& b+0 & \text{by A4.} \\ a &=& b & \text{by A3.} \end{array}$$

**3.1(ii)** $a \cdot 0 = 0$.

$$\begin{array}{rcll} a\cdot 0 &=& a(0+0) & \text{A3} \\ a\cdot 0 &=& a\cdot 0 + a\cdot 0 & \text{DL} \\ a\cdot 0 + 0 &=& a\cdot 0 + a\cdot 0 & \text{A3} \\ 0 + a\cdot 0 &=& a\cdot 0 + a\cdot 0 & \text{A2} \\ 0 &=& a\cdot 0 & \text{3.1(i)} \end{array}$$

**3.1(iii)** $(-a)b = -(ab)$.

$$\begin{array}{rcll} 0\cdot b &=& 0 & \text{3.1(ii)} \\ (a+(-a))b &=& 0 & \text{A4} \\ ab+(-a)b &=& 0 & \text{DL} \\ ab+-(ab) &=& 0 & \text{A4} \\ ab+-(ab) &=& ab+(-a)b & \text{since 0=0} \\ -(ab) &=& (-a)b & \text{A2 \& 3.1(i)} \end{array}$$

**3.1(iv)** $(-a)(-b) = ab$.

It will be useful to first prove that $-(-x) = x$. Since $(-x) + x = 0$ and $(-x) + (-(-x)) = 0$, Theorem 3.1(i) implies that $-(-x) = x$.

$$\begin{array}{rcll} (-a)(-b) &=& -(a(-b)) & \text{3.1(iii)} \\ &=& -((-b)a) & \text{M2} \\ &=& -(-(ba)) & \text{3.1(iii)} \\ &=& ab & \text{since } -(-x) = x \,\&\, \text{M2} \end{array}$$

**3.1(v)** $ac = bc$ & $c \neq 0 \implies a = b$.

First, $c^{-1}$ exists by M4.

$$\begin{array}{rcll} (ac)c^{-1} & = & (bc)c^{-1} & \\ a(cc^{-1}) & = & b(cc^{-1}) & \text{M1} \\ a \cdot 1 & = & b \cdot 1 & \text{M4} \\ a & = & b & \text{M3} \end{array}$$

**3.1(vi)** $ab = 0 \implies a = 0$ or $b = 0$.

Let $ab = 0$ and suppose $a \neq 0$ and $b \neq 0$. Thus, $a^{-1}$ and $b^{-1}$ exist.

$$\begin{array}{rcll} ab(b^{-1}) & = & 0 \cdot b^{-1} & \\ a(bb^{-1}) & = & 0 & \text{M1 \& 3.1(ii)} \\ a \cdot 1 & = & 0 & \text{M4} \\ a & = & 0 & \text{M3} \end{array}$$

But $a = 0$ contradicts our supposition. Thus, if $ab = 0$ then $a = 0$ or $b = 0$.

**3.2(i)** $a \leq b \implies -b \leq -a$.

$$\begin{array}{rcll} a & \leq & b & \text{given} \\ a + ((-a) + (-b)) & \leq & b + ((-a) + (-b)) & \text{O4} \\ (a + (-a)) + (-b) & \leq & b + ((-b) + (-a)) & \text{A1 \& A2} \\ 0 + (-b) & \leq & (b + (-b)) + (-a) & \text{A4 \& A1} \\ -b & \leq & 0 + (-a) & \text{A3 \& A4} \\ -b & \leq & -a & \text{A3} \end{array}$$

**3.2(ii)** $a \leq b$ & $c \leq 0 \implies bc \leq ac$.

I'll need to use that $-0 = 0$ so I'll prove that first.

$$\begin{array}{rcl} (i) & 0 + (-0) = 0 & \text{by A4.} \\ (ii) & 0 = 0 + 0 & \text{by A3.} \\ (i)\&(ii) & \implies 0 + (-0) = 0 + 0. & \\ \text{Thus,} & -0 = 0 & \text{by Thm 3.1(i).} \end{array}$$

Assume $a \leq b$ and $c \leq 0$. Since $c \leq 0$, 3.2(i) implies $-0 \leq -c$. Thus, $0 \leq -c$.

$$\begin{array}{rcll} a(-c) & \leq & b(-c) & \text{O5} \\ -(ac) & \leq & -(bc) & \text{M2,3.1(iii),M2} \\ bc & \leq & ac & \text{3.2(i)} \end{array}$$

**3.2(iii)** $0 \leq x$ & $0 \leq y \implies 0 \leq xy$.

Apply O5 with $a = 0$, $b = x$ and $c = y$.

**3.2(iv)** $0 \leq a^2$.

Either $a \leq 0$ or $0 \leq a$ by O1.

Suppose $0 \leq a$. Then $0 \leq a^2$ by O5 and Thm 3.1(ii).

Suppose $a \leq 0$. By 3.2(i) we have $0 \leq -a$. Thus, $0 \leq (-a)^2$.

Since $(-a)^2 = a^2$ by 3.1(iv) we have $0 \leq a^2$.