Limit Definitions

These are the formal definitions of each type of limit for functions of real numbers. Under each is the corresponding informal definition. You might look them over and see why they make sense. Draw a picture illustrating each type of limit. Assume L and a are real numbers.

1. $\lim_{x\to a} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x)-L| < \epsilon$. It is assumed the domain of f contains a set of the form $(a-\mu, a)\cup(a, a+\mu)$ for some $\mu > 0$.
   - As x approaches a from either side $f(x)$ approaches L.
2. $\lim_{x\to a^+} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $0 < x-a < \delta$ implies $|f(x)-L| < \epsilon$. It is assumed the domain of f contains a set of the form $(a, a+\mu)$ for some $\mu > 0$.
   - As x approaches a from the right side $f(x)$ approaches L.
3. $\lim_{x\to a^-} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $0 < a-x < \delta$ implies $|f(x)-L| < \epsilon$. It is assumed the domain of f contains a set of the form $(a-\mu, a)$ for some $\mu > 0$.
   - As x approaches a from the left side $f(x)$ approaches L.
4. $\lim_{x\to\infty} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, M > 0$ such that $x > M$ implies $|f(x)-L| < \epsilon$. It is assumed the domain of f contains a set of the form (b, ∞) for some $b \in \mathbb{R}$.
   - As x grows positively without bound $f(x)$ approaches L.
5. $\lim_{x\to-\infty} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, M < 0$ such that $x < M$ implies $|f(x)-L| < \epsilon$. It is assumed the domain of f contains a set of the form $(-\infty, b)$ for some $b \in \mathbb{R}$.
   - As x grows negatively without bound $f(x)$ approaches L.
6. $\lim_{x\to a} f(x) = \infty$ means $\forall\, B > 0\ \exists\, \delta > 0$ such that $0 < |x-a| < \delta$ implies $f(x) > B$. It is assumed the domain of f contains a set of the form $(a-\mu, a)\cup(a, a+\mu)$ for some $\mu > 0$.
   - As x approaches a from either side $f(x)$ grows positively without bound.
7. $\lim_{x\to a^+} f(x) = \infty$ means $\forall\, B > 0\ \exists\, \delta > 0$ such that $0 < x-a < \delta$ implies $f(x) > B$. It is assumed the domain of f contains a set of the form $(a, a+\mu)$ for some $\mu > 0$.
   - As x approaches a from the right side $f(x)$ grows positively without bound.
8. $\lim_{x\to a^-} f(x) = \infty$ means $\forall\, B > 0\ \exists\, \delta > 0$ such that $0 < a-x < \delta$ implies $f(x) > B$. It is assumed the domain of f contains a set of the form $(a-\mu, a)$ for some $\mu > 0$.
   - As x approaches a from the left side $f(x)$ grows positively without bound.
9. $\lim_{x\to a} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, \delta > 0$ such that $0 < |x-a| < \delta$ implies $f(x) < B$. It is assumed the domain of f contains a set of the form $(a-\mu, a)\cup(a, a+\mu)$ for some $\mu > 0$.
   - As x approaches a from either side $f(x)$ grows negatively without bound.
10. $\lim_{x\to a^+} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, \delta > 0$ such that $0 < x-a < \delta$ implies $f(x) < B$. It is assumed the domain of f contains a set of the form $(a, a+\mu)$ for some $\mu > 0$.
    - As x approaches a from the right side $f(x)$ grows negatively without bound.
11. $\lim_{x\to a^-} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, \delta > 0$ such that $0 < a-x < \delta$ implies $f(x) < B$. It is assumed the domain of f contains a set of the form $(a-\mu, a)$ for some $\mu > 0$.
    - As x approaches a from the left side $f(x)$ grows negatively without bound.
12. $\lim_{x\to\infty} f(x) = \infty$ means $\forall\, B > 0\ \exists\, M > 0$ such that $x > M$ implies $f(x) > B$. It is assumed the domain of f contains a set of the form (b, ∞) for some $b \in \mathbb{R}$.
    - As x grows positively without bound $f(x)$ grows positively without bound.
13. $\lim_{x\to\infty} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, M > 0$ such that $x > M$ implies $f(x) < B$. It is assumed the domain of f contains a set of the form (b, ∞) for some $b \in \mathbb{R}$.
    - As x grows positively without bound $f(x)$ grows negatively without bound.
14. $\lim_{x\to-\infty} f(x) = \infty$ means $\forall\, B > 0\ \exists\, M < 0$ such that $x < M$ implies $f(x) > B$. It is assumed the domain of f contains a set of the form $(-\infty, b)$ for some $b \in \mathbb{R}$.
    - As x grows negatively without bound $f(x)$ grows positively without bound.
15. $\lim_{x\to-\infty} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, M < 0$ such that $x < M$ implies $f(x) < B$. It is assumed the domain of f contains a set of the form $(-\infty, b)$ for some $b \in \mathbb{R}$.
    - As x grows negatively without bound $f(x)$ grows negatively without bound.