Summary of the properties of limits of functions of a real variable

Finite Limit Laws. Let c and a be a real numbers (constants). Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and are finite. Then the following hold.

(1) $\lim_{x\to a} c = c.$

(2) $\lim_{x\to a} x = a.$

(3) $\lim_{x\to a} c\,f(x) = c\,\lim_{x\to a} f(x).$

(4) $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x).$

(5) $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right).$

(6) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0.$

(7) $\lim_{x\to a} (f(x))^c = \left(\lim_{x\to a} f(x)\right)^c$, unless $\lim_{x\to a} f(x) = 0$ and $c < 0.$

Analogous statements are true if we replace $x \to a$ with $x \to a^+$, $x \to a^-$ or $x \to \pm\infty$.

Infinite Limit Laws. Let a, $L \neq 0$ and $c \neq 0$ be real constants. Let p,q, n, z, and k functions such that $\lim_{x\to a} p(x) = \infty$, $\lim_{x\to a} q(x) = \infty$, $\lim_{x\to a} z(x) = 0$ and $\lim_{x\to a} k(x) = L$. Then the following hold.

(1) $\lim_{x\to a} p(x) + q(x) = \infty$

(2) $\lim_{x\to a} p(x) \pm k(x) = \infty$

(3) $\lim_{x\to a} -p(x) \pm k(x) = -\infty$

(4) $\lim_{x\to a} p(x)q(x) = \infty$

(5) $\lim_{x\to a} -p(x)q(x) = -\infty$

(6) $\lim_{x\to a} cp(x) = \operatorname{sign}(c)\,\infty$

(7) $\lim_{x\to a} k(x)p(x) = \operatorname{sign}(L)\,\infty$

(8) $\lim_{x\to a} \frac{1}{p(x)} = 0$

(9) No conclusion can be drawn for $\lim_{x\to a} p(x)z(x)$ or $\lim_{x\to a} p(x) - q(x)$.

Analogous statements are true if we replace $x \to a$ with $x \to a^+$, $x \to a^-$ or $x \to \pm\infty$.

The infinite limit laws may abbreviated as follows.

(1) $\infty + \infty = \infty$

(2) $\infty \pm L = \infty$

(3) $-\infty \pm L = -\infty$

(4) $\infty \cdot \infty = \infty$

(5) $-\infty \cdot \infty = -\infty$

(6&7) $c\,\infty = \operatorname{sign}(c)\,\infty$

(8) $\frac{1}{\infty} = 0$

The Removable Singularity Rule. Suppose $g(x)$ is continuous on (a, c) and that $f(x) = g(x)$ on an $(a, b) \cup (b, c)$. Then $\lim_{x\to b} f(x) = g(b)$.

The Composition Theorem. If $\lim_{x\to a} g(x) = P$ and $\lim_{y\to P} f(y) = L$ then $\lim_{x\to a} f(g(x)) = L$. This holds true when any of a, P or L are infinities. If f is continuous at P then this can be written as

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(L).$$

The Squeeze Theorem. Suppose $f(x) \leq g(x) \leq h(x)$ on a suitable domain. Then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x)$, provided the limits exist. If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = L$. This holds when $L = \pm\infty$ and for limits as $x \to \pm\infty$ or one sided limits. (The reader should be able to determine what is meant by a *suitable domain*.)