Limits of Functions
Section 20

Discuss textbook's Definition 20.1. It is awkward because it is combining many cases (15 cases). See handout with all 15 cases. Yikes!

Standard Definition. Let a and L be real numbers. Then

$$\lim_{x \to a} f(x) = L$$

means

$$\forall \epsilon > 0, \exists \delta > 0 \, s.t. 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

It is assumed the domain of f contains a set of the form $(a - \mu, a) \cup (a, a + \mu)$ for some $\mu > 0$ and that x is in the domain of f.

Note. $\{x \in \mathbb{R} \,|\, 0 < |x - a| < \delta\} = (a - \delta, a) \cup (a, a + \delta)$. Such a set is sometimes called a **deleted neighborhood** of a.

Theorem. Assume a, K and L are real numbers and that f and g are real valued functions with a common domain that contains a set of the form $(a - \mu, a) \cup (a, a + \mu)$ for some $\mu > 0$.

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$, then $\lim_{x \to a} f(x) + g(x) = L + K$.

Proof. Let $\epsilon > 0$. Assume x is always in the domain of f and g.

$\exists\ \delta_1 > 0$ s.t. $0 < |x - a| < \delta_1$ implies $|f(x) - L| < \epsilon/2$.

$\exists\ \delta_2 > 0$ s.t. $0 < |x - a| < \delta_2$ implies $|g(x) - K| < \epsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - a| < \delta$ we have

$$|f(x) + g(x) - (L + K)| \leq |f(x) - L| + |g(x) - K| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Definition. Let a be a real number and f be a real valued function whose domain contains a set of the form $(a-\mu, a) \cup (a, a+\mu)$ for some $\mu > 0$. Then

$$\lim_{x \to a} f(x) = \infty$$

means

$$\forall B > 0, \exists \delta > 0 \, s.t. 0 < |x-a| < \delta \implies f(x) > B.$$

Theorem. Let a and L be real numbers. Let f and g be real valued functions with a common domain that contains a set of the form $(a-\mu, a) \cup (a, a+\mu)$.

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f(x) + g(x) = \infty$.

Proof. Assume x is in the domain of f and g.
Let $\epsilon >$ and $B > 0$.
$\exists\ \delta_1 > 0$ s.t. $0 < |x-a| < \delta_1$ implies $f(x) > B + |L| + \epsilon$.
$\exists\ \delta_2 > 0$ s.t. $0 < |x-a| < \delta_2$ implies $|g(x) - L| < \epsilon$.
Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x-a| < \delta$ we have

$$f(x) + g(x) > B + |L| + \epsilon + L - \epsilon \geq B.$$

□

Sequence Characterization of Limits of Functions

Theorem. Let a and L be real numbers. $\lim_{x \to a} f(x) = L$ $\Leftrightarrow$ ($a_n \to a$ implies $\lim_{n \to \infty} f(a_n) \to L$). This assumes $a_n \neq a$ and all a_n are in the domain of f.

Proof. ($\Rightarrow$) Suppose, $\lim_{x \to a} f(x) = L$. Let $a_n \to a$. We must show $f(a_n) \to L$. Let $\epsilon > 0$.
$\exists\ \delta > 0$ s.t. $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$.
$\exists\ N$ s.t. $n > N$ implies $|a_n - a| < \delta$
Thus, $n > N$ implies $|f(a_n) - L| < \epsilon$. Hence, $f(a_n) \to L$.

($\Leftarrow$) Assume that $a_n \to a$ implies $f(a_n) \to L$. Suppose however that it is false that $\lim_{x \to a} f(x) = L$. We shall derive a contradiction.

$\exists\, \epsilon > 0$ s.t. $\forall\, \delta > 0\; \exists\; x \in (a - \delta, a) \cup (a, a + \delta)$ (and x in domain of f) s.t. $|f(x) - L| \geq \epsilon$..

Fix such an $\epsilon > 0$. $\forall n \in \mathbb{N}\; \exists\; x_n \in (a-1/n, a) \cup (a, a+1/n)$ (and x_n is in domain of f) s.t $|f(x_n) - L| \geq \epsilon$.

Then $x_n \to a$, but $f(x_n)$ does not converge to L. Contradiction!

□

Each of the 15 limit definitions has an equivalent reformulation in terms of sequences.

Theorem. Let a, L and K be real numbers and let f and g be real valued functions with suitable domains.
If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$, then $\lim_{x \to a} f(x)g(x) = LK$.

Before reading the proof go back and reread the proof of Theorem 9.4 which stated that $\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$ when both exist.

Proof. Let $a_n \to a$. Assume each a_n is in the domain of f and g and no $a_n = a$. Then $\lim_{n \to \infty} f(a_n) = L$ and $\lim_{n \to \infty} g(a_n) = K$. Thus, by Theorem 9.4, $\lim_{n \to \infty} f(a_n)g(a_n) = LK$. Then by the theorem just proved above we have

$$\lim_{x \to a} f(x)g(x) = LK.$$

□