More Term by Term
Sections 25 & 26

We return to some theorems about the integration of series of functions.

Theorem [25.2] Let (f_n) be a sequence of continuous real valued functions on $[a, b]$. Assume $f_n \to f$ uniformly on $[a, b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Before doing the proof we give two examples showing the conclusion can fail if the assumes do not hold.

Example 1. Let $\{q_1, q_2, q_3, \ldots\}$ be an enumeration of the rational numbers in $[0, 1]$. Let $f_n : [0, 1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} 1 & \text{for } x \in \{q_1, q_2, \ldots, q_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to f$, pointwise, where

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

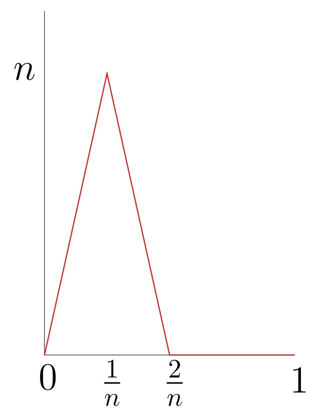
But,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 0 = 0,$$

while

$$\int_0^1 f(x)\,dx \quad \text{does not exist.}$$

Example 2. Let $f_n(x)$ be defined on [0,1] by the graph below.



Now each f_n is continuous and $f_n \to 0$, pointwise, on $[0,1]$.

But,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 1 = 1,$$

while

$$\int_0^1 0\,dx = 0.$$

Proof of Theorem 25.2. The functions, $f_n - f$, are continuous and hence intergable on $[a.b]$.

Let $\epsilon > 0$.

By uniform convergence $\exists\ N$ s.t. $n > N \implies$

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}, \quad \forall x \in [a,b].$$

Thus, for $n > N$ we have

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b f_n(x) - f(x)\,dx\right| \leq$$

$$\int_a^b |f_n(x) - f(x)|\, dx < \int_a^b \frac{\epsilon}{b-a}\, dx = \epsilon.$$

Thus,

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

□

Theorem [26.4] Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R. Then

$$\int_0^x f(t)\, dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1},$$

has the same radius of convergence.

The proof follows from 25.2, see page 210.

However, derivatives do not behave well.

Example. Let $f_n(x) = \frac{1}{n}\sin nx$. Then $f_n \to 0$, uniformly on $\mathbb{R}$. Yet, $f_n'(x) = \cos nx$ does not converge.

However, derivatives of power series are well behaved. See Theorem 26.5.