Integration by Parts
The Product Rule in Reverse

Unfortunately there is no systematic procedure for integrating the product of two functions with known anti-derivatives. However, the product rule for derivatives does yield a useful trick for doing certain integrals. The method is called **Integration by Parts.** Let u and v be functions of x; but we will also think of them as variables in their own rights. The product rule states,

$$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrate both sides.

$$\int \frac{duv}{dx}\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx.$$

By the Substitution Theorem,

$$\int duv = \int u\,dv + \int v\,du,$$

or,

$$uv + C = \int u\,dv + \int v\,du.$$

This is normally rewritten as

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

(We drop the C since both sides have indefinite integrals.)

What is this good for? We do some examples.

Example 1. Find $\displaystyle\int x\cos x\,dx$.

Solution. Let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ and $v = \sin x$. Now

$$\begin{aligned}\int x\cos x\,dx &= \int u\,dv\\ &= uv - \int v\,du\\ &= x\sin x - \int \sin x\,dx\\ &= x\sin x + \cos x + C\end{aligned}$$

$\square$

We did not need to set $v = \sin x + C$ since all that we require is the v be some anti-derivative of $\cos x$. The key is making the right choices for u and v. We used $u = x$ since the derivative of x is simpler than x and the integral of $\cos x$ does not get any "worse". Try redoing Example 1 with $u = \cos x$ and $dv = dx$ and see what happens. However, there is not a hard and fast rule for how to chose u and dv.

Example 2. Find $\int \ln x \, dx$.

Solution. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. Thus,

$$\begin{aligned} \int \ln x \, dx &= \int u \, dv \\ &= uv - \int v \, du \\ &= x \ln x - \int x \frac{1}{x} \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \end{aligned}$$

□

Example 3. Find $\int e^x \sin x \, dx$.

Solution. This will be a two step process. Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$. Thus,

$$\begin{aligned} \int e^x \sin x \, dx &= -\int u \, dv \\ &= -uv + \int v \, du \\ &= -e^x \cos x + \int e^x \cos x \, dx \qquad (1) \end{aligned}$$

But now what to do? We apply integration by parts to the last integral. This, as you will check, gives

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

By substituting into equation (1) we get,

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx.$$

(Now for a trick!) This last equation implies,

$$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x + C,$$

or

$$\int e^x \sin x\, dx = \frac{e^x(\sin x - \cos x)}{2} + C.$$

You may need to reread this example several times, working out the steps on your own. □

Example 4. Find $I = \int \sec^3 x\, dx$.

Solution. It is not obvious how to start. Since the integral $\sec^2 x$ is easy we will try $u = \sec x$ and $dv = \sec^2\, dx$. Then $du = \sec x \tan x\, dx$ and $v = \tan x$. Thus,

$$I = \int \sec x\ \sec^2 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec^3 x - \sec x\, dx.$$

Thus,

$$I = \sec x \tan x - I + \ln|\sec x + \tan x| + C.$$

And finally,

$$I = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C.$$

□

Example 5. Find $\int x^2 \sin x\, dx$.

Solution. Taking a derivative of x^2 makes it simpler and integrating the sine function won't make matters worse. So, we let $u = x^2$ and $dv = \sin x\, dx$. Then $du = 2xdx$ and $v = -\cos x$. Thus,

$$\int x^2 \sin x\, dx = \int u\, dv = uv - \int vdu = -x^2 \cos x + 2\int x \cos x\, dx.$$

This is a step in the right direction. Now let $u = x$ and $dv = \cos x\, dx$. Then $du = dx$ and $v = \sin x$. Thus,

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

Combining these results we get

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

You can check the result by taking the derivative. □

Example 6. Find $\int x^3 e^x\, dx$.

Solution. Let $u = x^3$ and $dv = e^x\,du$. Then $du = 3x^2\,dx$ and $v = e^x$. Thus,

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx.$$

We repeat. Let $u = x^2$ and $dv = e^x\,du$. Then $du = 2x\,dx$ and $v = e^x$. Thus,

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

Once more. Let $u = x$ and $dv = e^x\,du$. Then $du = dx$ and $v = e^x$. Thus,

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Therefore,

$$\int x^3 e^x\,dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C.$$

□

This leads to the idea of a **reduction formula**.

Example 7. Show that $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$. This is an example of a reduction formula.

Solution. Let $u = x^n$ and $dv = e^x\,dx$. Then $du = nx^{n-1}$ and $v = e^x$. Therefore,

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx.$$

□

Example 8. Find $\int x^6 e^x\,dx$.

Solution.

$$\begin{aligned}
\int x^6 e^x\,dx &= x^6 e^x - 6\left(\int x^5 e^x\,dx\right)\\
&= x^6 e^x - 6x^5 e^x + 6\cdot 5\left(\int x^4 e^x\,dx\right)\\
&= x^6 e^x - 6x^5 e^x + 6\cdot 5x^4 e^x - 6\cdot 5\cdot 4\left(\int x^3 e^x\,dx\right)\\
&= x^6 e^x - 6x^5 e^x + 6\cdot 5x^4 e^x - 6\cdot 5\cdot 4x^3 e^x + 6\cdot 5\cdot 4\cdot 3\left(\int x^2 e^x\,dx\right)\\
&= x^6 e^x - 6x^5 e^x + 6\cdot 5x^4 e^x - 6\cdot 5\cdot 4x^3 e^x + 6\cdot 5\cdot 4\cdot 3x^2 e^x - 6\cdot 5\cdot 4\cdot 3\cdot 2\left(\int x e^x\,dx\right)\\
&= x^6 e^x - 6x^5 e^x + 6\cdot 5x^4 e^x - 6\cdot 5\cdot 4x^3 e^x + 6\cdot 5\cdot 4\cdot 3x^2 e^x - 6\cdot 5\cdot 4\cdot 3\cdot 2x e^x + 6!e^x + C.
\end{aligned}$$

□

By inductionn we can now show that

$$\int x^n e^x \, dx = (-1)^n e^x \sum_{k=1}^{n} (-1)^k \frac{n! x^k}{k!} + C.$$

We conclude with a few pratice problems.

(1) $\int x \sin 3x \, dx$
(2) $\int x^2 e^x \, dx$
(3) $\int x \ln x \, dx$
(4) $\int x^2 \ln x \, dx$
(5) $\int e^x \cos x \, dx$
(6) $\int e^{2x} \cos 3x \, dx$
(7) $\int e^{ax} \sin bx \, dx$
(8) $\int \arctan x \, dx$
(9) $\int \sqrt{x} \ln x \, dx$
(10) $\int x \csc^2 x \, dx$
(11) Find reducetion formulas for $\int x^n \sin x \, dx$ and $\int x^n \cos x \, dx$. Find $\int x^{11} \sin x \, dx$.