Power Series
Chapter 4
Section 23

Let $f_n : D \to \mathbb{R}$ for $n = 0, 1, 2, 3, \ldots$ be functions. We study infinite sequences and series of functions.

For now we just do **power series** which are series of functions of the form

$$\sum_{n=p}^{\infty} a_n x^n$$

where usually $p = 0$ or 1.

(If all the a_n were the same value we would have a geometric series.)

You may recall from Calculus that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x,$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x.$$

Theorem. Consider $\sum_{n=p}^{\infty} a_n x^n$. Let

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Then R is called the **radius of convergence** and

(i) the series converges if $|x| < R$,

(ii) the series diverges if $|x| > R$.

No conclusion can be drawn when $|x| = R$; one has to *check the end points.* Note that if $R = \infty$ the series converges for all x. The series always converges for $x = 0$.

The proof uses the Root Test. The version of the Root Test we did in class used only the limit of $\sqrt[n]{|a_n|}$ rather than the limit superior done in the textbook. The proof both cases is the same. The reason we need the stronger version will be clear when we do examples.

Proof. Case 1. Suppose $0 < R < \infty$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \frac{|x|}{R}.$$

For $|x| < R$ the series converges by the Root Test and for $|x| > R$ the series diverges by the Root Test.

Case 2. Suppose $R = 0$. Then for $x \neq 0$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \infty.$$

Hence Root Test implies the series diverges.

Case 3. Suppose $R = \infty$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = 0 < 1$$

and the Root Test implies the series converges. $\square$

In problems where the limit of the ratio a_{n+1}/a_n exists it is equal to $\sqrt[n]{|a_n x^n|}$ by Corollary 12.3, so you can just use the Ratio Test. We will see in the examples below how we can sometimes still find R when this fails.

Example. Find the radius of convergence of $\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$.

Solution. We will show that the radius of convergence is ∞.

Notice that the odd powers of x are missing. Write it out as

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = 1+0x-\tfrac{1}{2!}x^2+0x^3+\tfrac{1}{4!}x^4+0x^5-\tfrac{1}{6!}x^6+0x^7+\tfrac{1}{8!}x^8\cdots$$

This is equal to the power series $\sum_{n=0}^{\infty} a_n x^n$, where

$$a_n = \begin{cases} 0 & \text{for } n \text{ odd} \\ \frac{(-1)^{n/2}}{n!} & \text{for } n \text{ even} \end{cases}$$

Since every odd term has $a_n = 0$, we have $\liminf \sqrt[n]{|a_n x^n|} = 0$. The $\limsup$ is the limit of the even terms.

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{k\to\infty} \sqrt[2k]{|a_{2k} x^{2k}|} = |x| \lim_{n\to\infty} \frac{1}{\sqrt[2k]{(2k)!}}.$$

Now this is a subsequence of $\left(\frac{1}{\sqrt[n]{(n)!}}\right)$. This limit is similar to Exercise 12.14a. It uses Corollary 12.3. Thus,

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(n)!}} = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0.$$

Thus, $R = \infty$. □

Example. (This is Example 6 from the textbook.) Find the radius of convergence of $\sum_{k=0}^{\infty} 2^{-k} x^{3k}$.

Solution I. Write it out as a power series.

$$\sum_{k=0}^{\infty} 2^{-k} x^{3k} = 1+0x+0x^2+\tfrac{1}{2}x^3+0x^4+0x^5+\tfrac{1}{4}x^6+0x^7+0x^8+\tfrac{1}{8}x^9+0x^{10}\cdots$$

Thus $a_n = 0$ whenever $n \bmod 3 = 1$ or 2. For $n \bmod 3 = 0$ we have $a_n = 2^{-n/3}$. Thus,

$$\limsup \sqrt[n]{|a_n|} = \limsup \sqrt[n]{2^{-n/3}} = 2^{-1/3}.$$

Thus, $R = 2^{1/3} \approx 1.25992$. $\square$

Solution II. Let $y = x^3$ and consider the series $\sum_{k=0}^{\infty} 2^{-k} y^k$. Show it converges for $|y| < 2$. Thus, the original series has a radius of convergence of $2^{1/3}$. $\square$