Sequences and Limits

Sequences.

Definition. A **finite sequence** of real numbers is a function from an ordered finite set, *e.g.* $\{1, 2, 3, \dots n\}$, into $\mathbb{R}$. It can also be thought of as a ordered subset of $\mathbb{R}$. When listing the members we use parentheses, (), instead of brackets, {}. For example,

$$(0, 4, 7, 7, -3, 0.34, \pi, 0, 0, 1)$$

is a finite sequence. Note that repeated elements are allowed. The **underlying set** for this finite sequence is

$$\{\pi, 0, 4, 7, -3, 0.34\},$$

in no particular order. (I'll give the formal definition of underlying set later.)

Definition. A **countably infinite sequence** of real numbers is a function from $\mathbb{N}$ into $\mathbb{R}$. The domain $\mathbb{N}$ can be replaced by sets like $\{0\} \cup \mathbb{N}$, $\{3, 4, 5, \dots\}$, or { the even positive integers }. The domain has to have a least member and each member must have a successor.

If there is an apparent pattern we can list the first few members of a sequence and then end with "…" . Again we use parentheses.

$$(1, 1, 2, \tfrac{1}{2}, 3, \tfrac{1}{3}, 4, \tfrac{1}{4}, \dots)$$

$$(2, 3, 5, 7, 11, 13, 17, \dots)$$

It may be that the pattern does not start right away.

$$(\sqrt{5}, 0, 0, \tfrac{1}{3}, -10, 17, -\tfrac{3}{20}, e, \pi, e, \pi, e, \pi, \dots)$$

Sometimes we can find or are given a formula for the terms of a sequence. For Example, let $S = (s_n)_{i=1}^{\infty}$ where $s_n = \frac{n}{n+2}$. Then

$$S = (\tfrac{1}{3}, \tfrac{2}{4}, \tfrac{3}{5}, \tfrac{4}{6}, \dots).$$

Sometimes the formula is **recursive.** Example, the **Fibonacci sequence** is

$$(f_n)_{n=1}^{\infty} = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots),$$

which is defined by letting $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all integers $n \geq 3$. Such recursive sequences often arise in finding series solution to differential equations. This is covered in MATH 305. See Chapter 5 of *Elementary Differential Equations & Boundary Value Problems* by Boyce & DiPrima.

Definition. The **underlying set** of a sequence is its range as a subset of $\mathbb{R}$. Said differently, the underlying set of a sequence (s_n) is the set $\{s_n \,|\, n \in \mathbb{N}\}$. For example, the underlying set of the sequence

$$(1, -1, 1, -1, 1, -1, 1, -1, ...)$$

is the set $\{1, -1\}$.

Some More Definitions.

- A sequence is **bounded above** if its underlying set is bounded above.
- A sequence is **bounded below** if its underlying set is bounded below.
- A sequence is **bounded** if its underlying set is bounded above and below.
- A sequence (s_n) is **increasing** if $s_n < s_{n+1}$ $\forall n \in \mathbb{N}$.
- A sequence (s_n) is **decreasing** if $s_n > s_{n+1}$ $\forall n \in \mathbb{N}$.
- A sequence (s_n) is **nondecreasing** if $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$.

- A sequence (s_n) is **nonincreasing** if $s_n \geq s_{n+1}\ \forall n \in \mathbb{N}$.
- A sequence that is increasing, decreasing, nonincreasing or nondecreasing is said to be **monotone**.
- A sequence is **eventually** (increasing, decreasing, nonincreasing, nondecreasing or monotone) if there is a natural number N such that the sequence is (increasing, decreasing, nonincreasing, nondecreasing or monotone) for all $n \geq N$.

Optional Comments. Functions from $\mathbb{Z}$ into $\mathbb{R}$ are called **bi-infinite sequences**. These are used in information theory. See *An Introduction to Symbolic Dynamics and Coding* by Marcus & Lind. There is a generalization of sequences called **nets** that involve functions from (potentially) uncountable sets into $\mathbb{R}$. See *General Topology* by Willard.

Limits.

Definition. A sequence $(s_n)_{n=1}^{\infty}$ **converges** to a **limit** $L \in \mathbb{R}$ if

$$\forall \epsilon > 0\, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |s_n - L| < \epsilon.$$

The notation for this is $\lim_{n\to\infty} s_n = L$ or simply $s_n \to L$.

Example 1. $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. Let $\epsilon > 0$ be given. By the Archimedean Property $\exists\, N \in \mathbb{N}$ such that $1/N < \epsilon$. If $n \geq N$ then $1/n \leq 1/N$. Thus,

$$\left|\tfrac{1}{n} - 0\right| < \epsilon.$$

Therefore, $\lim_{n\to\infty} \frac{1}{n} = 0$. $\square$

Example 2. $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$.

Scratch Work. Let $\epsilon > 0$ be given. We want to get n big enough that

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \epsilon.$$

Set up as below and solve for n.

$$-\epsilon < \frac{n}{2n+1} - \frac{1}{2} < \epsilon$$

$$-\epsilon(2n+1) < n - \frac{2n+1}{2} < \epsilon(2n+1)$$

$$-\epsilon(2n+1) < -\frac{1}{2} < \epsilon(2n+1)$$

In the last line the right hand inequality is true for natural numbers n. By the Archimedean Property there is a natural number N such that $N\epsilon > 1/2$. For $n \geq N$ we have $2n+1 > N$ so

$$-\epsilon(2n+1) < -\epsilon N < -\frac{1}{2}.$$

Now that we see what to do we write out the proof. □

Proof. Let $\epsilon > 0$ be given. By the Archimedean Property the is a natural number N such that $N\epsilon > 1/2$. Then for $n \geq N$ we have $2n+1 > N$.

$$-\epsilon(2n+1) < -\epsilon N < -\frac{1}{2} < \epsilon(2n+1).$$

$$-\epsilon(2n+1) < n - n - \frac{1}{2} < \epsilon(2n+1)$$

$$-\epsilon(2n+1) < n - \frac{2n+1}{2} < \epsilon(2n+1)$$

$$-\epsilon < \frac{n}{2n+1} - \frac{1}{2} < \epsilon$$

Thus, $\forall\, n \geq N$ we have

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \epsilon.$$

Hence, $\lim\limits_{n\to\infty} \dfrac{n}{2n+1} = \dfrac{1}{2}$.

□

Example 3. $\lim\limits_{n\to\infty} \dfrac{1}{n^2+1} = 0.$

Proof. $\forall\, n \in \mathbb{N}$ we know $1 \leq n$, which implies $n \leq n^2$ and thus, $n < n^2 + 1$. Therefore,

$$\frac{1}{n^2+1} < \frac{1}{n}.$$

Let $\epsilon > 0$. $\exists\, N \in \mathbb{N}$ such that $n \geq N$ implies $\frac{1}{n} < \epsilon$ by Example 1. Thus, $\forall\, n \geq N$

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n} < \epsilon.$$

Hence, $\lim\limits_{n\to\infty} \dfrac{1}{n^2+1} = 0.$ □

It occurs to us that the idea in this example could be generalized into a useful tool.

Theorem 1. Let (a_n) and (b_n) be sequence. If $0 \leq a_n \leq b_n$ for all n and $b_n \to 0$, then $a_n \to 0$.

Proof. Let $\epsilon > 0$. $\exists\ N \in \mathbb{N}$ such that $n \geq N$ implies $|b_n - 0| < \epsilon$. Thus,

$$|a_n - 0| = a_n \leq b_n = |b_n - 0| < \epsilon.$$

Thus, $\lim\limits_{n\to\infty} a_n = 0.$ □

Example 4. Prove that $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

Proof. Since $n \geq 1$, we know $n^2 \geq n$. Hence, $n^3 \geq n^2$ and so $n^3 \geq n$. Thus,

$$\frac{1}{n^3} \leq \frac{1}{n}.$$

Since, we know from Example 1 that $1/n \to 0$, Theorem 1 tells us that $\lim_{n\to\infty} \frac{1}{n^3} = 0$. $\square$

Example 5. Let $p \in \mathbb{N}$. Prove that $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Proof. By induction you can prove that $n^p \geq n$. Hence $1/n^p \leq 1/n$. The result follows from Theorem 1. $\square$

Example 6. Prove that $\lim_{n\to\infty} \frac{1}{n!} = 0$.

Proof. We claim that $n! \geq n$. This is true for $n = 1$ since $1! = 1$. If for a given $k \geq 1$ we have $k! \geq k$ then $(k+1)! = (k+1)k! \geq (k+1)k \geq k+1$. By induction $n! \geq n$ for all natural numbers n. Thus,

$$\frac{1}{n!} \leq \frac{1}{n}.$$

By applying Theorem 1 we have $\lim_{n\to\infty} \frac{1}{n!} = 0$. $\square$

Example 7. Prove that $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

Proof. We will use induction to prove that $2^n \geq n$ for all $n \in \mathbb{N}$. For $n = 1$ we have $2^1 \geq 1$. Assume $2^k \geq k$ for some given $k \geq 1$. Then

$$2^{k+1} = 2 \cdot 2^k \geq 2k \geq k+1,$$

where the last inequality follows from $k \geq 1$.

Now we have $\frac{1}{2^n} \leq \frac{1}{n}$ so Theorem 1 gives the result. $\square$

Example 8. Prove that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Our trick will not work. We will do a direct proof. We will need the following fact which we prove first: if $0 < a < b$ then $\sqrt{a} < \sqrt{b}$. *Proof.* We use proof by contradiction. Suppose

$$\sqrt{a} \geq \sqrt{b}.$$

Then

$$a = \sqrt{a}\sqrt{a} \geq \sqrt{a}\sqrt{b}.$$

But also,

$$\sqrt{a}\sqrt{b} \geq \sqrt{b}\sqrt{b} = b.$$

Combining these last two lines gives $a \geq b$, which contradicts the given fact that $a < b$. Therefore, if $0 < a < b$ then $\sqrt{a} < \sqrt{b}$.

Now we return to Example 8.

Proof. Let $\epsilon > 0$. $\exists \quad N \in \mathbb{N}$ such that $N > 1/\epsilon^2$. Let $n \geq N$. Then $\sqrt{n} > 1/\epsilon$. Thus,

$$\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} < \epsilon.$$

Thus, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. $\square$

Example 9. Prove that the sequence $(1, -1, 1, -1, 1, -1, \dots)$ does not converge.

Proof. This seems obvious, but the proof is an exercise in logic. Let $s_n = (-1)^{n+1}$. The claim that (s_n) does not have a limit is equivalent to the following. For every $L \in \mathbb{R}$

$\exists\, \epsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists n \geq N$, such that $|s_n - L| \geq \epsilon$.

Suppose $s_n \to L$. Either $L \geq 0$ or $L < 0$.

Case 1: Assume $L \geq 0$. Let $\epsilon = 0.1$. Let N be any natural number. Let n be an even natural number greater than N. (For example we could let $n = 2N$.) Then $s_n = -1$. We have

$$|s_n - L| = |-1 - L| = L + 1 > 0.1,$$

since L is nonnegative. Therefore, L is not the limit of (s_n).

Case 2: Assume $L < 0$. Let $\epsilon = 0.1$. Let N be any natural number. Let n be an odd natural number greater than N. (For example we could let $n = 2N + 1$.) Then $s_n = 1$. We have

$$|s_n - L| = 1 + (-L) > 0.1,$$

since $-L$ is positive. Therefore, L is not the limit of (s_n).

The inescapable conclusion is that (s_n) does not have a limit. □

Example 10. Let $a_n = n^2$. Then (a_n) does not converge to any real number.

Proof. Suppose $a_n \to L \in \mathbb{R}$. Let $\epsilon = 0.01$. By the Archimedean Property there is an $N \in \mathbb{N}$ such that $N > |L + 0.5|$. Then for $n \geq N$ we have

$$|a_n - L| = |n^2 - L| = n^2 - L > (L + 0.5)^2 - L = L^2 + 0.25 \geq 0.25 > 0.01.$$

Therefore, (n^2) does not converge to a real number. □

In Section 9 of the textbook several standard theorems about sequences are developed. I'll state these here. We will go over some of the proofs in class. You should be able to read (unpack and understand) all of these proofs. The textbook skips one important result: if $a_n = K$ for all n, then $a_n \to K$. This is obvious, but try to write out a formal proof for yourself.

Theorem 9.1. If a sequence converges to a real number then it is bounded.

Theorem 9.2. Let $k \in R$ and suppose $s_n \to s \in \mathbb{R}$. Then the sequence (ks_n) converges to ks. That is, $\lim_{n\to\infty} ks_n = k \cdot \lim_{n\to\infty} s_n$, when the second limit exists.

Theorem 9.3. If $s_n \to s$ and $t_n \to t$, then $(s_n + t_n) \to s + t$. That is, $\lim_{n\to\infty} s_n + t_n = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$, when the limits on the right exist.

Theorem 9.4. If $s_n \to s$ and $t_n \to t$, then $(s_n t_n) \to st$. That is, $\lim_{n\to\infty} s_n t_n = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n)$, when the limits on the right exist.

Theorem 9.5. If $t_n \to t$ where $t_n \neq 0 \quad \forall n$ and $t \neq 0$, then $1/t_n \to 1/t$.

Theorem 9.6. If $s_n \to s$ and $t_n \to t$, where $t_n \neq 0 \, \forall n$ and $t \neq 0$ then $s_n/t_n \to s/t$.

The proofs of the next theorem are pretty difficult. Treat them as optional reading.

Theorem 9.7 (Basic Examples).
(a) $\lim_{n\to\infty} 1/n^p = 0$ for $p > 0$. (The book's proof is not valid.)
(b) $\lim_{n\to\infty} a^n = 0$ for $|a| < 1$.
(c) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(d) $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for $a > 0$.

Using these one can do many examples very quickly.

Example. Prove that $\lim_{n\to\infty} \frac{5n^2+n+3}{3n^2+7} = \frac{5}{3}$.

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \frac{5n^2+n+3}{3n^2+7} &= \lim_{n\to\infty} \frac{5+\frac{1}{n}+\frac{3}{n^2}}{3+\frac{7}{n^2}} \\
&= \frac{\lim_{n\to\infty} 5 + \lim_{n\to\infty} \frac{1}{n} + 3 \lim_{n\to\infty} \frac{1}{n^2}}{\lim_{n\to\infty} 3 + 7 \lim_{n\to\infty} \frac{1}{n^2}} \\
&= \frac{5+0+3\cdot 0}{3+7\cdot 0} \\
&= \frac{5}{3}
\end{aligned}$$

□

Proof of Theorem 9.3. Let $s_n \to s$ and $t_n \to t$. We need to show $s_n + t_n \to s + t$.

Proof. Let $\epsilon > 0$.

$\exists\, N_1 \in \mathbb{N}$ s.t. $n \geq N \implies |s_n - s| < \epsilon/2$.

$\exists\, N_2 \in \mathbb{N}$ s.t. $n \geq N \implies |t_n - t| < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$ we have

$|(s_n+t_n)-(s+t)| = |(s_n-s)+(t_n-t)| \leq |s_n-s|+|t_n-t| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$

Thus, $s_n + t_n \to s + t$. $\square$

Proof of Theorem 9.4. Let $s_n \to s$ and $t_n \to t$. We need to show $s_n t_n \to st$. This turns out to be a bit harder. We will need to use the fact that Theorem 9.1 says convergent sequences are bounded.

Proof. Let $\epsilon > 0$. By Theorem 9.1 $\exists\ M > 0$ s.t. $|s_n| \leq M$ $\forall\ n \in \mathbb{N}$.

$\exists\ N_1 \in \mathbb{N}$ s.t. $n \geq N \implies |t_n - t| < \epsilon/2M$.

$\exists\ N_2 \in \mathbb{N}$ s.t. $n \geq N \implies |s_n - s| < \epsilon/(2|t| + 1)$.

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$ we have

$$\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&\leq |s_n t_n - s_n t| + |s_n t - st| \\
&= |s_n||t_n - t| + |t||s_n - s| \\
&< M\tfrac{\epsilon}{2M} + |t|\tfrac{\epsilon}{2|t|+1} \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Thus, $s_n t_n \to st$. $\square$

About the proof of Theorem 9.7 (a). The claim is $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$ for all $p > 0$. The textbook notes that n^p has only been defined for p rational. Thus, the proof given (page 48) is only valid for that case. The textbook's proof uses a fact that has not been shown: For $p > 0$, $0 < a < b \implies a^p < b^p$. We will show this for $p \in \mathbb{Q} \cap (0, \infty)$.

We have shown this for $p \in \mathbb{N}$ and $p = 1/2$. Let $p = 1/k$ for some $k \in \mathbb{N}$. Suppose, $b^{\frac{1}{k}} \leq a^{\frac{1}{k}}$. Then

$$\left(b^{\frac{1}{k}}\right)^k \leq \left(b^{\frac{1}{k}}\right)^k \implies b \leq a,$$

which is false. Now, let $p = \frac{m}{k}$ for natural numbers m and k. Then

$$0 < a < b \implies a^m < b^m \implies a^{\frac{m}{k}} < b^{\frac{m}{k}}.$$

Proof of Theorem 9.7 (c). The claim is $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. This will be used several times in this course. The textbook's proof is a little sketchy, so here we will fill in the details. The proof uses the Binomial Theorem. This is an exercise on page 6 is Ross's textbook. You may have seen the proof in MATH 349. The Binomial Theorem says

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof. Since $n \geq 1$, we know $n^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$. Let $s_n = n^{\frac{1}{n}} - 1 \geq 0$. Notice $n = (1 + s_n)^n$. We have

$$(1 + s_n)^n = 1 + ns_n + \tfrac{n(n-1)}{2} s_n^2 + \cdots + s_n^n.$$

Assume $n \geq 2$. Then,

$$(1 + s_n)^n > \tfrac{n(n-1)}{2} s_n^2.$$

Thus,

$$n > \tfrac{n(n-1)}{2} s_n^2.$$

This implies $\frac{2}{n-1} > s_n^2$, which implies $s_n < \sqrt{\frac{2}{n-1}}$. Now, you can show that

$$\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0.$$

Thus, $\lim_{n\to\infty} s_n = 0$. Hence, $\lim_{n\to\infty} n^{\frac{1}{n}} - 1 = 0$. The desired result follows. $\square$

Sequences with Infinite Limits

Definition. Let (s_n) be an infinite sequence of real numbers.

- $\lim_{n\to\infty} s_n = \infty$ if for every positive real number B there is a natural number N such that $n > N$ implies $s_n > B$.
- $\lim_{n\to\infty} s_n = -\infty$ if for every negative real number B there is a natural number N such that $n > N$ implies $s_n < B$.

Example. Prove that $\lim_{n\to\infty} n^2 = \infty$.

Proof. Let $B > 0$ be given. There exists a natural number N larger than B. Let $n > N$. Then $n^2 > N^2 > B^2$. If $B > 1$ we have $B^2 > B$, and so $n^2 > B$. If $B \in (0, 1]$, we still have $n^2 > B$ since $n^2 > 1$. Thus, $\lim_{n\to\infty} n^2 = \infty$. □

Example. Prove that $\lim_{n\to\infty} \dfrac{n^2}{n+1} = \infty$.

Proof. Let $B > 0$ be given. Notice that $\frac{n^2}{n+1} = n - 1 + \frac{1}{n+1}$. Let N be a natural number greater than $B + 1$. Then for $n > N$ we have

$$n - 1 + \frac{1}{n+1} > n - 1 > N - 1 > B.$$

Thus, $\lim_{n\to\infty} \dfrac{n^2}{n+1} = \infty$. □

There are analogs of some of the theorems for finite limits to infinite limits. The statements are a bit more complex. I'll just list a few of them here.

- If $s_n \to \infty$ and $t_n \to \infty$, then $s_n + t_n \to \infty$.
- If $s_n \to \infty$ and $t_n \to L \in \mathbb{R}$, then $s_n + t_n \to \infty$.

- If $s_n \to \infty$ and (t_n) is bounded, then $s_n + t_n \to \infty$.
- If $s_n \to \infty$ and $t_n \to \infty$, then $s_n t_n \to \infty$.
- If $s_n \to \infty$ and $t_n \to L > 0$, then $s_n t_n \to \infty$.
- If s_n is never zero and $s_n \to \infty$, then $1/s_n \to 0$.
- If $s_n \to \infty$ and $t_n > s_n$, then $t_n \to \infty$.

Similar statements hold for limits to $-\infty$. However, if $s_n \to \infty$ and $t_n \to -\infty$, no general conclusion can be drawn about the limit of $(s_n + t_n)$. If $s_n \to \infty$ and $t_n \to 0$, no general conclusion can be drawn about the limit of $(s_n t_n)$.

Here is an application.

Theorem. Let $p(x) = a_m x^m + \cdots + a_1 x + a_0$ and $q(x) = b_k x^k + \cdots + b_1 x + b_0$ be polynomials with real coefficients; assume $a_m \neq 0$ and $b_k \neq 0$.

- If $m > k$, then $\lim_{n\to\infty} \dfrac{p(n)}{q(n)} = \pm\infty$, where the sign is the same as the sign of a_m/b_k.
- If $m < k$, then $\lim_{n\to\infty} \dfrac{p(n)}{q(n)} = 0$.
- If $m = k$, then $\lim_{n\to\infty} \dfrac{p(n)}{q(n)} = \dfrac{a_m}{b_k}$.