Series

(Sections 14 & 15)

Definition. The **infinite sum** is defined to

$$\sum_{k=p}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=p}^{n} a_k$$

when the limit exists. Usually the sum will start with $k = 1$ or 0.

The **n-th partial sum** of $\displaystyle\sum_{k=p}^{\infty} a_k$ is $s_n = \displaystyle\sum_{k=p}^{p+n-1} a_k$.

Now we can write $\displaystyle\sum_{k=p}^{\infty} a_k = \lim_{n\to\infty} s_n$ when the limit exists.

Note. If we leave off the first few terms of a sequence the limit is unaffected. This is a special case of the subsequence theorem. If we leave off a finite number of terms of an infinite sum, then whether it converges or not is unaffected, but if it converges the value will be changed.

A series converges if and only if the sequence of partial sums is Cauchy. Notice that for $n > m$

$$s_n - s_m = \sum_{k=m+1}^{n} a_n.$$

We say a series satisfies the **Cauchy criteria** if $\forall\ \epsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t.

$$n \geq m > N \implies \left| \sum_{k=m}^{n} a_k \right| < \epsilon.$$

Thus, a series converges if and only if it satisfies the Cauchy criteria. (This is Theorem 14.4 in the textbook.)

Corollary (14.5). If $\sum a_k$ converges then $a_k \to 0$. Proof. Let $\epsilon > 0$. $\exists\, N \in \mathbb{N}$ s.t.

$$n \geq m > N \implies \left|\sum_{k=m}^{n} a_k\right| < \epsilon.$$

Thus, for $k > N$ we have $|a_k| < \epsilon$ by using $n = m = k$. Thus, $a_k \to 0$.

Example. (This is Exercise 4.14 in your textbook.) $\displaystyle\sum_{n=1}^{\infty} \tfrac{1}{n} = \infty.$

Proof. Let $s_m = \displaystyle\sum_{n=1}^{m} \tfrac{1}{n}$. We will show $s_m \to \infty$. Consider the subsequence $(s_{m_k})_{k=1}^{\infty}$ where $m_k = 2^k$.

We will use induction to prove that $s_{m_k} \geq 1 + \frac{k}{2}$ for all $k \in \mathbb{N}$. First notice that $s_{m_1} = s_2 = 1 + \frac{1}{2}$. Suppose, $s_{m_{k-1}} \geq 1 + \frac{k-1}{2}$ for some $k > 1$. Now,

$$s_{m_k} = \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{2^{k-1}}\right) + \left(\tfrac{1}{2^{k-1}+1} + \cdots + \tfrac{1}{2^k}\right).$$

Each pair of parentheses has 2^{k-1} terms. In fact the sum in the first pair is $s_{m_{k-1}}$. Each term in the second pair before the last term is greater than the last term. Thus,

$$s_{m_k} \geq s_{m_{k-1}} + 2^{k-1} \cdot \left(\tfrac{1}{2^k}\right) = s_{m_{k-1}} + \tfrac{1}{2}.$$

By the induction hypothesis $s_{m_{k-1}} \geq 1 + \frac{k-1}{2}$. Thus,

$$s_{m_k} \geq 1 + \tfrac{k-1}{2} + \tfrac{1}{2} = 1 + \tfrac{k}{2}.$$

Since $1 + k/2 \to \infty$ we have that $s_{m_k} \to \infty$. Thus (s_m) cannot converge. To show it diverges to infinity we use the fact that (s_m) is increasing since all the $\frac{1}{n}$ are positive. Let

$B > 0$. $\exists\ K \in \mathbb{N}$ s.t.

$$k \geq K \implies s_{m_k} > B.$$

Thus,

$$m \geq m_k \implies s_m \geq s_{m_k} > B.$$

Thus, $s_m \to \infty$. □

Definitions. A series $\sum a_k$ is called **alternating** if a_{k+1} always has the opposite sign of a_k. They come up in many applications. Consider a series $\sum a_n$. If $\sum |a_n|$ converges we say the series is **absolutely convergent**. We will see below that if $\sum |a_n|$ converges then so does $\sum a_n$. If $\sum a_n$ converges but $\sum |a_n|$ does not we say the series is **conditionally convergent.**

There is a slew of tests for convergence. We will run through these and prove a few of them.

The Direct Comparison Test. (14.6) Assume $a_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $\sum a_n$ converges (to a finite real number) and $|b_n| \leq a_n\ \forall n \in \mathbb{N}$, then $\sum b_n$ converges (to a finite real number $\leq \sum a_n$).
(b) If $\sum a_n = \infty$ and $b_n \geq a_n\ \forall n \in \mathbb{N}$, then $\sum b_n = \infty$.
(c) Both (a) and (b) hold true if "$\forall n \in \mathbb{N}$" is replaced by "$\forall n > K$ for some $K \in \mathbb{N}$".

Proof of (a). Let $\epsilon > 0$. Let N be s.t. $n \geq m > N$ implies $\displaystyle\sum_{k=m}^{n} a_k < \epsilon$. Then

$$\left|\sum_{k=m}^{n} b_k\right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k < \epsilon.$$

You do (b) and (c). □

Note. A special case of (a) is that if $\sum |a_n|$ converges then so does $\sum a_n$.

Theorem. Assume $\sum a_n$ and $\sum b_b$ converge and let $c \in \mathbb{R}$. Then
(i) $\sum ca_n = c \sum a_n$, and
(ii) $\sum a_n + b_n = \sum a_n + \sum b_n$.

Proofs are easy and left to you. Note that if $\sum a_n$ diverges and $c \neq 0$, then $\sum ca_n$ diverges too.

The Limit Comparison Test. (Not in your textbook.) Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0, \infty),$$

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof. Let $\epsilon = L/2$. Then $\exists\ N$ s.t.

$$n > N \implies \frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}.$$

Thus,

$$\frac{Lb_n}{2} < a_n < \frac{3Lb_n}{2}.$$

If $\sum b_n$ converges, then so do $\sum Lb_n/2$ and $\sum 3Lb_n/2$. Thus, $\sum a_n$ converges by the Direct Comparison Test (c).

If a positive term series diverges, it diverges to infinity.
If $\sum b_n$ diverges to ∞ , then so does $\sum Lb_n/2$. By the Direct Comparison Test $\sum a_n$ diverges too.

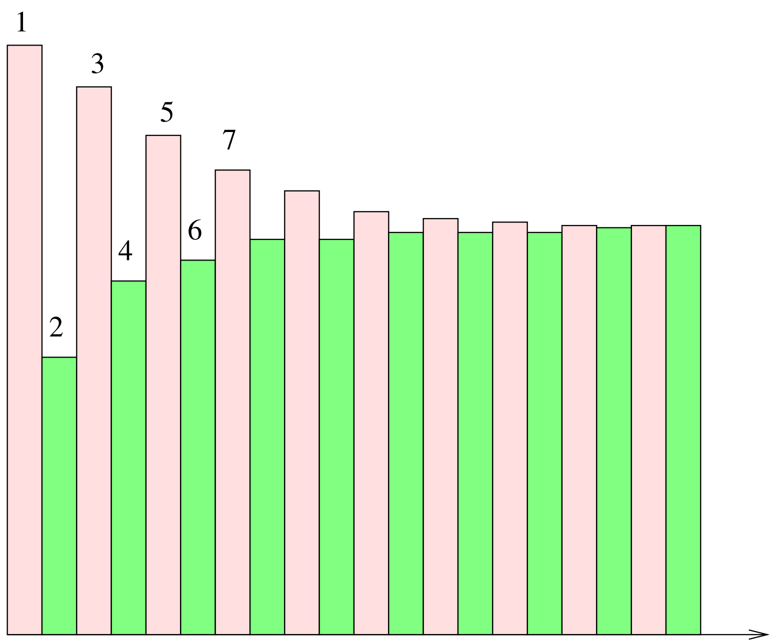
The other two implications follow logically from these or note that $\lim_{n\to\infty} b_n/a_n = 1/L \in (0, \infty)$. □

Alternating Series Test. Assume a_n is nonnegative and nonincreasing for all n; that is

$$0 \leq a_{n+1} \leq a_n, \ \ \forall n \in \mathbb{N}.$$

If $a_n \to 0$, then the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Idea of the proof. See the figure below. It is a bar graph of the partial sums, $s_m = \displaystyle\sum_{n=1}^{m}(-1)^{n+1}a_n$. Notice that the subsequence of odd terms is decreasing and bounded below while the even terms are increasing and bounded above. I'll discuss it in class.



□

Proof. Let $s_m = \sum_{n=1}^{m}(-1)^{n+1}a_n$. Consider the subsequences (s_{2k}) and (s_{2k-1}). We will show that following.

(*) The (green) subsequence (s_{2k}) is nondecreasing and bounded above. Hence it has a limit L_e.

(**) The (pink) subsequence (s_{2k-1}) is nonincreasing and bounded below. Hence it has a limit L_o.
(***) Finally, we show $L_e = L_o$ and that this is the limit of (s_m).

(*) $s_{2(k+1)} - s_{2k} = -a_{2k+2} + a_{2k+1} \geq 0$ implies $s_{2(k+1)} \geq s_{2k}$. Thus, (s_{2k}) is nondecreasing. We claim $s_{2k} \leq a_1$ for all k. We write

$$s_{2k} = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots - a_{2k-2} + a_{2k-1} - a_{2k}$$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2k-2} - a_{2k-1}) - a_{2k}.$$

All the terms in parentheses are positive or zero as is the last term. Hence, $s_{2k} \leq a_1$.

(**) $s_{2(k+1)-1} - s_{2k-1} = a_{2k+1} - a_{2k} \leq 0$ implies $s_{2(k+1)-1} \leq s_{2k-1}$. Thus, (s_{2k-1}) is nonincreasing. We claim $s_{2k-1} \geq a_1 - a_2$ for all k. We write

$$s_{2k-1} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \cdots a_{2k-3} - a_{2k-2} + a_{2k-1}$$

$$= a_1 - a_2 + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2k-3} - a_{2k-2}) + a_{2k-1}.$$

All the terms in parentheses are positive or zero as is the last term. Hence, $s_{2k-1} \geq a_1 - a_2$.

(***) We compute $L_e - L_o =$

$$\lim_{k\to\infty} s_{2k} - \lim_{k\to\infty} s_{2k-1} = \lim_{k\to\infty} s_{2k} - s_{2k-1} = \lim_{k\to\infty} -a_{2k} = 0.$$

Hence $L_e = L_o$. Let $L = L_e$.

Let $\epsilon > 0$. $\exists N$ s.t. for $k > N$ we have $|s_{2k} - L| < \epsilon$ and $|s_{2k-1} - L| < \epsilon$. Hence for and $n > 2N$ we have $|s_n - L| < \epsilon$. Thus $s_n \to L$ which is to say

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = L.$$

□

Examples.

The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges. It can be shown that the limit is $\ln 2$.

The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ converges. It can be shown that the limit is $\pi^2/12$.

Geometric Series Test. A series of the form $\sum ar^k$ is called a **geometric series**.

(i) If $r \neq 1$, then

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}.$$

(ii) If $|r| < 1$, then

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Proof. You have done this. □

We did some examples in class.

Ratio Test. [This version is the one given in most calculus textbooks. Your textbook gives a jazzed up version that is covered in 452.] Let $\sum a_n$ be an infinite series of nonzero terms.

(i) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, then $\sum a_n$ diverges.

Proof. The idea is to use apply the Direct Comparison Theorem using a geometric series.

(i) $\exists\, r \in \mathbb{R}$ s.t. $L < r < 1$. Let $\epsilon = r - L$. Now, $\exists\, N \in \mathbb{N}$ s.t. $n \geq N$ implies

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon.$$

Thus,

$$-(r - L) < \left|\frac{a_{n+1}}{a_n}\right| - L < r - L.$$

Thus,

$$\left|\frac{a_{n+1}}{a_n}\right| < r.$$

Thus,

$$|a_{n+1}| < |a_n| r, \ \ \forall n \geq N.$$

By induction,

$$|a_{N+k}| < |a_N| r^k, \ \ \forall k \in \mathbb{N}.$$

By the Geometric Series Test

$$\sum |a_N| r^k$$

converges since $|r| < 1$. By the Direct Comparison Test

$$\sum_{k=1}^{\infty} |a_{N+k}|$$

converges. Therefore,

$$\sum_{n=1}^{\infty} |a_n|$$

converges.

(ii) Similar. ☐

Example. The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Proof. We use the Ratio Test.

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \to 0 < 1.$$

□

Later we will show that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Root Test. [This version is the one given in most calculus textbooks. Your textbook gives a jazzed up version that is covered in 452.] Let $\sum a_n$ be an infinite series.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum a_n$ diverges.

Proof. (i) $\exists\, r \in \mathbb{R}$ s.t. $L < r < 1$. $\exists\, N \in \mathbb{N}$ s.t. $n \geq N$ implies

$$\left| \sqrt[n]{|a_n|} - L \right| < r - L.$$

Thus,

$$\sqrt[n]{|a_n|} < r.$$

Thus,

$$|a_n| < r^n.$$

Thus,

$$\sum_{n=N}^{\infty} |a_n|$$

converges by the Direct Comparison Test since $|r| < 1$. It follows that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

You can prove (ii). □

Integral Test. See textbook.

Proof. See textbook. □

p-Series Test. The infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges to ∞ for $p \leq 1$.

Proof. Use the Integral Test. See textbook. □

Examples.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. In fact it equals $\pi^2/6$.

See `https://en.wikipedia.org/wiki/Basel_problem`.

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges. In fact it equals $\pi^4/90$.

$\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges. In fact it equals $\pi^6/945$.

$\sum_{n=1}^{\infty} \frac{1}{n^8}$ converges. In fact it equals $\pi^8/9450$.

You might wonder about the odds powers. Look up the Riemann Zeta function.