Uniform Continuity
Section 19

Definition. Let $f : D \to \mathbb{R}$, Then f is **uniformly continuous** on D if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$x, y \in D \,\&\, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 7x + 5$. Then f is uniformly continuous on $\mathbb{R}$.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/7$. Suppose $|x - y| < \delta$. Then

$$\big|(7x + 5) - (7y + 5)\big| = 7|x - y| < 7\delta = \epsilon.$$

$\square$

Exercise. In the space below show that $g(x) = 14 - 3x$ is uniformly continuous on $\mathbb{R}$.

Example. Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = 1/x$. Then f is continuous on $(0, \infty)$ but not uniformly continuous on $(0, \infty)$.

Proof. We know from Theorem 17.4 (iii) that f is continuous on $(0, \infty)$. To show f is not uniformly continuous on $(0, \infty)$ we will establish the **negation** of the definition:

$$\exists\, \epsilon > 0 \text{ s.t. } \forall \delta > 0,\ \exists\, x, y \in (0, \infty) \text{ with } |x - y| < \delta, \text{ but } |f(x) - f(y)| \geq \epsilon.$$

Let $\epsilon = 1$. Let $\delta > 0$.

Idea! I'll let $x = \delta$ and y be somewhere in between 0 and x. Clearly, $|x - y| < \delta$. I can then make $|1/x - 1/y|$ large by moving y near to 0. I want this gap to be bigger than 1, so I'll use $1/y = 1/\delta + 3$. See figure below.

Let $x = \delta$ and $y = \frac{1}{\frac{1}{\delta}+3} = \frac{\delta}{1+3\delta}$. Then

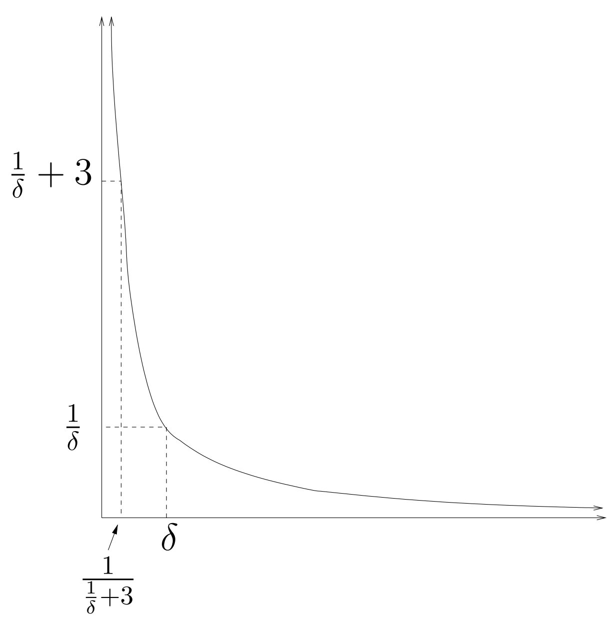
$$|x - y| = \left|\delta - \frac{\delta}{1+3\delta}\right| = \frac{3\delta^2}{1+3\delta} < \delta.$$

To see why the last $<$ is true notice: $0 < \delta \implies 3\delta^2 < \delta + 3\delta^2 \implies \frac{3\delta^2}{1+3\delta} < \delta$.

But,

$$|f(x) - f(y)| = \left|\frac{1}{\delta} - \left(\frac{1}{\delta} + 3\right)\right| = 3 > 1 = \epsilon.$$

$\square$



Example. We show that $f(x) = 1/x$ is uniformly continuous on $[1, \infty)$.

Proof. Let $\epsilon > 0$. Pick $\delta = \epsilon$. Suppose $|x - y| < \delta$. Then

$$\left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y - x}{xy} \right| < \frac{\epsilon}{xy} \leq \epsilon.$$

□

Exercise. In the space below prove that $f(x) = 1/x^2$ is uniformly continuous on $[4, \infty)$.

Exercise. In the space below prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, but is uniformly continuous on any bounded interval.

Theorem (19.2). If f is continuous on $[a, b]$, then it is uniformly continuous on $[a, b]$.

Proof. Suppose f is continuous, but not uniformly continuous on $[a, b]$. Then

$$\exists\, \epsilon > 0 \text{ s.t. } \forall\, \delta > 0,\ \exists\, x, y \in [a, b] \text{ with } |x - y| < \delta, \text{ but } |f(x) - f(y)| \geq \epsilon.$$

Fix such an $\epsilon > 0$.

Then $\forall\, n \in \mathbb{N}$, $\exists\ x_n, y_n \in [a, b]$ with $|x_n - y_n| < 1/n$, but $|f(x_n) - f(y_n)| \geq \epsilon$.

Consider the sequence, (x_n). By the Bolzano-Weierstrass Theorem there is a convergent subsequence. Suppose $x_{n_k} \to c \in [a, b]$.

We claim the corresponding subsequence of (y_n) converges to the same value, that is $y_{n_k} \to c$. *Proof.* Let $\eta > 0$. (That is the Greek letter eta.) Let K be s.t. $k > K$ implies $|x_{n_k} - c| < \eta/2$ and $1/n_k < \eta/2$. Then

$$|y_{n_k} - c| = |y_{n_k} - x_{n_k} + x_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < \eta/2 + \eta/2 = \eta.$$

Since f is continuous $f(x_{n_k}) \to f(c)$ and $f(y_{n_k}) \to f(c)$. Thus,

$$|f(x_{n_k}) - f(y_{n_k})| \to |f(c) - f(c)| = 0.$$

But, this is impossible since for all k we know

$$|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon.$$

□

Recall. We defined f to be continuous at x if $x_n \to x \implies f(x_n) \to f(x)$. We also know that convergent sequences are Cauchy and vice versa. So, you might think the continuous image of a Cauchy sequence would be Cauchy. But this is not true. Let $f(x) = 1/x$ on $(0, \infty)$. The sequence $x_n = 1/n$ is Cauchy, but $f(x_n) = n$ is certainly not

Cauchy. Of course the problem is f is not continuous at 0. However, we do have the following theorem.

Theorem (19.4). Let f be uniformly continuous on D. Let (x_n) be a Cauchy sequence in D. Then $(f(x_n))$ is Cauchy.

Proof. Let $\epsilon > 0$.
$\exists\, \delta > 0$ s.t. $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
$\exists\, N$ s.t. $m, n > N$ implies $|x_n - x_m| < \delta$.
Thus, $m, n > N$ implies $|f(x_n) - f(x_m)| < \epsilon$. □

Definition. Let $f : A \to \mathbb{R}$ and $A \subset B$. If $g : B \to \mathbb{R}$ is s.t. $g(a) = f(a)\ \forall\, a \in A$, then g is an **extension** of f. If f is continuous and g is also continuous, then g is a **continuous extension** of f.

Example. Let $f(x) = \frac{\sin x}{x}$ on $\mathbb{R} - \{0\}$. Let $g : \mathbb{R} \to \mathbb{R}$ be

$$g(x) = \begin{cases} f(x) & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0. \end{cases}$$

It can be shown that g is continuous on $\mathbb{R}$, hence g is a continuous extension of f.

Example. Let $f(x) = 1/x$ on $\mathbb{R} - \{0\}$. Then there is no continuous extension of f on $\mathbb{R}$.

Example. Let $f(x) = \frac{x^3+3x^2-4x-12}{x^2-4}$ on $(-2, 2)$. Does f have a continuous extension on $[-2, 2]$?

Answer. Yes! Notice, $f(x) = x + 3$. Define $g(x)$ on $[-2, 2]$ to be $x + 3$. Then g is a continuous extension of f. □

Theorem (19.5). Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous on (a, b). Then f has a unique continuous extension on $[a, b]$.

Outline of Proof. 1. Let (a_n) be a sequence in (a, b) that converges at a. Show $(f(a_n))$ converges and let $y_a = \lim_{n\to\infty} f(a_n)$.

2. Likewise, let $y_b = \lim_{n\to\infty} f(b_n)$ for a sequence (b_n) in $[a, b]$ that converges to b.

3. Define

$$g(x) = \begin{cases} y_a & \text{for} \quad x = a \\ f(x) & \text{for} \quad x \in (a, b) \\ y_b & \text{for} \quad x = b. \end{cases}$$

4. Show that g is continuous on $[a, b]$.

5. Show that any other choice for the values for $g(x)$ at $x = a$ and $x = b$ would not give a continuous function. $\square$