Uniform Convergence of Functions
Section 24

Let (f_n) be a sequence of real valued functions all with domain $D \subset \mathbb{R}$. Let f be a real valued function with domain $D \subset \mathbb{R}$. We want to define what it means for (f_n) to converge to f. There are two different definitions that are used, **pointwise convergence** and **uniform convergence**.

It will turn out the uniform convergence implies pointwise convergence, but not the reverse. It will also turn out the the uniform limit of continuous functions is continuous, but this is not true for pointwise convergence.

Definition. We say (f_n) **converges pointwise** to f on D if

$$\forall x \in D \,\&\, \epsilon > 0,\ \exists\, N \text{ s.t. } n > N \implies |f_n(x) - f(x)| < \epsilon.$$

This is exactly the same as saying $f_n(x) \to f(x)$ for each $x \in D$. When this happens we write

$$\lim_{n\to\infty} f_n = f \text{ or } f_n \to f.$$

Definition. We say (f_n) **converges uniformly** to f on D if

$$\forall \epsilon > 0,\ \exists N \text{ s.t. } \forall\, x \in D,\ n > N \implies |f_n(x) - f(x)| < \epsilon.$$

When this happens we write

$$\lim_{n\to\infty} f_n = f \text{ or } f_n \to f.$$

Some books use $f_n \rightrightarrows f$.

Fact. It is immediate that uniform convergence implies pointwise convergence.

Example. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in [0,1) \\ 1 & \text{for} \quad x = 1. \end{cases}$$

Then f_n convergences pointwise to f, but not uniformly.

Theorem. If $f_n \rightrightarrows f$ on D and the f_n functions are continuous on D, then f is continuous on D.

Proof. Let $x \in D$. We will prove f is continuous at x. Let $\epsilon > 0$. We observe that for any $n \in \mathbb{N}$

$$|f(x)-f(y)| = |f(x)-f_n(x)+f_n(x)-f_n(y)+f_n(y)-f(y)| \leq$$
$$|f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|.$$

Now $\exists\, n$ s.t. $|f(x)-f_n(x)| < \epsilon/3$ and $|f(y)-f_n(y)| < \epsilon/3$.

Fix such an n. Now $\exists \delta > 0$ s.t. $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon/3$, since f_n is continuous (assuming $y \in D$).

Thus, $y \in D$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. Hence, f is continuous at x. Since x was an arbitrary point in D we have that f is continuous on D. Note: we are not claiming f in uniformly continuous on D. □

There are many examples in the textbook. Be sure to study them. We will do a few here.

Example. Let $f_n(x) = \frac{\arctan(x^n + \sin nx)}{n}$, for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that (f_n) converges uniformly to 0 on $\mathbb{R}$.

Solution. We know that $-\frac{\pi}{2} < \arctan(\text{anything}) < \frac{\pi}{2}$.

Thus, we always have $|f_n(x) - 0| < \frac{\pi}{2n}$.

Let $\epsilon > 0$. $\exists\ N$ s.t. $1/N < 2\epsilon/\pi$. Then for $n > N$ we have

$$|f_n(x) - 0| < \frac{\pi}{2n} < \epsilon,$$

$\forall\ x \in \mathbb{R}$.

Since, N did not depend on x, we have shown that (f_n) converges uniformly to 0 on $\mathbb{R}$. □

Example. Let $f_n(x) = x^2/n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that (f_n) converges to 0 pointwise on $\mathbb{R}$, but not uniformly.

Solution. Fix $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \tfrac{x^2}{n} = x^2 \lim_{n\to\infty} \tfrac{1}{n} = x^2 \cdot 0 = 0.$$

Thus, (f_n) converges to 0 pointwise on $\mathbb{R}$.

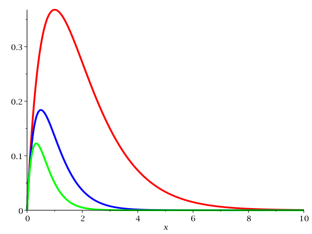
Now, to show the convergence is not uniform we pick $\epsilon = 1$. Fix any natural number n, $\exists\ x \in \mathbb{R}$ s.t. $x > \sqrt{n}$. Thus,

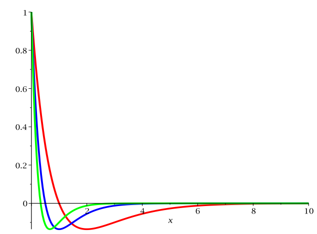
$$\left|\tfrac{x^2}{n} - 0\right| = \tfrac{x^2}{n} > 1.$$

Hence, the convergence is not uniform over $\mathbb{R}$. □

Question. In the last example what happens if the domain is a bounded set instead of $\mathbb{R}$?

Example. Let $f_n(x) = xe^{-nx}$ and $g_n(x) = f_n'(x)$ for all $n \in \mathbb{N}$ and $x \geq 0$. Show that (f_n) converges uniformly to 0 on $[0, \infty)$, but that (g_n) does not converge uniformly on $[0, \infty)$; find the pointwise limit of (g_n). The graphs for $f_1(x)$, $f_2(x)$, $f_3(x)$, $g_1(x)$, $g_2(x)$ and $g_3(x)$ are below





Solution. First we consider the pointwise limit of $(f_n(x))$. It is obviously 0 for $x = 0$. Fix a value for $x \in (0, \infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} xe^{-nx} = x \lim_{n\to\infty} (e^{-n})^x = x \left(\lim_{n\to\infty} \frac{1}{e^n}\right)^x = x(0)^x = 0.$$

Thus, the pointwise limit is 0.

To show uniform convergence we notice that the graphs of f_n have a global maximum over the domain. If for any $\epsilon > 0$ we can find an N such the the peak is less than ϵ, that N will work for all $x \geq 0$. Now,

$$f_n'(x) = (1 - nx)e^{-nx}.$$

The maximum occurs at $x = 1/n$ where $f_n(1/n) = 1/en$. (You can use the second derivative test if you want to check that this is local maximum. Since, there are no other critical points, $f_n(0) = 0$ and, using L'Hopital's rule you can show $f_n(x) \to 0$ as $x \to \infty$, we have a global maximum on the domain.)

Let $\epsilon > 0$. $\exists$ N s.t. $1/N < \epsilon$. Let $n > N$. Then

$$|f_n x(x) - 0| \leq 1/ne < 1/n < \epsilon.$$

Thus, we have uniform convergence to 0 on $[0, \infty)$.

Now for (g_n). Clearly, $g_n(0) = 1$ for all n. From the graphs it appears the limit is zero for $x > 0$. You can use L'Hopital's rule to show that indeed, for a fixed $x > 0$,

$$\lim_{n\to\infty} (1 - nx)e^{-nx} = 0.$$

Thus, on $[0, \infty)$ we have that (g_n) converges pointwise to

$$g(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x > 0. \end{cases}$$

Since g is not continuous, the convergence is not uniform. $\square$

Example. Let $f_n(x) = \frac{nx+\sin^2 x}{3n+\cos^2 x}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, Show that (f_n) converges pointwise to $f(x) = x/3$. Is the convergence uniform?

Solution. First we consider the pointwise limit.

$$f_n(x) = \frac{x + \frac{\sin^2 x}{n}}{3 + \frac{\cos^2 x}{n}} \to \frac{x+0}{3+0} = \frac{x}{3},$$

where the limit is taken as $n \to \infty$ and x is fixed.

Next we will "try" to show the convergence is uniform and see what goes "wrong". Let $\epsilon > 0$.

$$|f_n(x) - f(x)| = \left| \frac{nx + \sin^2 x}{3n + \cos^2 x} - \frac{x}{3} \right|$$

$$= \left| \frac{x + \frac{\sin^2 x}{n}}{3 + \frac{\cos^2 x}{n}} - \frac{x}{3} \right| = \left| \frac{3x + 3\frac{\sin^2 x}{n} - 3x - x\frac{\cos^2 x}{n}}{3\left(3 + \frac{\cos^2 x}{n}\right)} \right|$$

$$= \left| \frac{3\frac{\sin^2 x}{n} - x\frac{\cos^2 x}{n}}{3\left(3 + \frac{\cos^2 x}{n}\right)} \right| = (*).$$

Notice, perhaps after some experimentation, that this is unbounded as x increases. Let $x = 12\pi n$. Then we get

$$(*) \geq \left| \frac{0 - 12\pi}{12} \right| = \pi.$$

Thus, if we choose any positive value for ϵ below π and any natural number n we could always find a value for x where $|f_n(x) - f(x)| > \epsilon$. $\square$