Integration by Substitution [1]

The Chain Rule in Reverse

We develop a powerful technique for finding indefinite integrals. Suppose F and g are known differentiable functions. Let $F' = f$, that is F is an anti-derivative of f. Let h be defined by

$$h(x) = f(g(x))g'(x),$$

where we assume $Range(g) \subset Dom(f)$. Suppose we want to find the anti-derivative of h. Then

$$\begin{aligned}
\int h(x)\,dx &= \int f(g(x))g'(x)\,dx \\
&= \int F'(g(x))g'(x)\,dx \\
&= \int [F(g(x))]'\,dx \qquad \text{(The Chain Rule backwards!)} \\
&= F(g(x)) + C \qquad \text{(The FTC)}
\end{aligned}$$

Example 1. Find $\int \cos(x^2)2x\,dx$.

Solution. Let $g(x) = x^2$, and $f(x) = \cos(x)$. Then use $F(x) = \sin(x)$ as an anti-derivative of $f(x)$.

$$\begin{aligned}
\int \cos(x^2)2x\,dx &= \int \cos(g(x))g'(x)\,dx \\
&= \int [\sin(g(x))]'\,dx \qquad \text{(The Chain Rule backwards!)} \\
&= \sin(g(x)) + C \qquad \text{(The FTC)} \\
&= \sin(x^2) + C
\end{aligned}$$

□

It is easy to check that $(\sin(x^2)+C)' = \cos(x^2)2x$. Of course you need to use the Chain Rule.

Note: If we had started with $\int x\cos(x^2)\,dx$ we could apply this method as follows.

$$\int x\cos(x^2)\,dx = \frac{1}{2}\int \cos(x^2)2x\,dx = \cdots = \frac{1}{2}\sin(x^2) + C.$$

Notice, we used just C and not $C/2$. Why is this okay?

Example 2. Find $\displaystyle\int \frac{x^2}{\sqrt{x^3+7}}\,dx$.

Solution. Spotting what to use for f and g is harder. Let $g(x) = x^3+7$. Then $g'(x) = 3x^2$. Thus we write,

$$\int \frac{x^2}{\sqrt{x^3+7}}\,dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{x^3+7}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{g(x)}}g'(x)\,dx.$$

Now let $f(x) = 1/\sqrt{x}$. Then use $F(x) = 2\sqrt{x}$ for an anti-derivative of $f(x)$. Then we have

$$\begin{aligned}
\frac{1}{3}\int \frac{3x^2}{\sqrt{x^3+7}}\,dx &= \frac{1}{3}\int f(g(x))g'(x)\,dx \\
&= \frac{1}{3}\int [F(g(x))]'\,dx \\
&= \frac{1}{3}F(g(x)) + C \\
&= \frac{2}{3}\sqrt{x^3+7} + C
\end{aligned}$$

□

This is getting pretty tough. We will do one more example and then develop a short cut! (Hang in there.)

Example 3. Find $\displaystyle\int \frac{e^x}{e^x+1}\,dx$.

Solution. Guessing at what to use for g and f is far from obvious. But the following will work, let $g(x) = e^x + 1$. Then $g'(x) = e^x$. Now let $f(x) = 1/x$ and use $F(x) = \ln|x|$. Then,

$$\begin{aligned}
\int \frac{e^x}{e^x+1}\,dx &= \int f(g(x))g'(x)\,dx \\
&= \int F'(g(x))g'(x)\,dx \\
&= \int [F(g(x))]'\,dx \\
&= F(g(x)) + C = \ln|e^x+1| + C = \ln(e^x+1) + C.
\end{aligned}$$

Check this by computing the derivative. □

A short cut! There has got to be a better way. We are going to redo Example 1, using somewhat different notation. Then we will point out a short cut that has been staring at us all along. We shall use u instead of $g(x)$ and think of u both as a function of x and as a variable in its own right.

Example 4. Find $\int \cos(x^2)2x\,dx$.

Solution. Let $u = x^2$. Then $\frac{du}{dx} = 2x$. Now we compute.

$$\begin{aligned}
\int \cos(x^2)2x\,dx &= \int \cos(u)\frac{du}{dx}\,dx && (1)\\
&= \int \frac{d\sin(u)}{dx}\,dx && (2)\\
&= \sin(u) + C = \sin(x^2) + C && (3)
\end{aligned}$$

But now notice,

$$\sin(u) + C = \int \cos(u)\,du, \qquad (4)$$

thinking of u as a variable.

□

The problem is that going from equation (1) to (2) is hard. But going from (1) directly to (4) is easy to remember:

$$\int \cos(u)\frac{du}{dx}\,dx = \int \cos(u)\,du.$$

The trick is to jump from (1) to (4) and then work back to (3). It is almost as though you could cancel the dx's. Leibniz designed the notation of calculus just to make tricks like this work. So, we could redo this example as follows.

$$\int \cos(x^2)2x\,dx = \int \cos(u)\frac{du}{dx}\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(x^2) + C$$

In fact it is common practice to abuse the notion and write "$du = 2x\,dx$". Then

$$\int \cos(x^2)2x\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(x^2) + C$$

Again, these are mnemonic tricks. They give mathematically correct answers because the math in equations (1)-(4) above is valid. This method is often referred to as *u-substitution.* We will formally state it as a theorem.

Theorem 1 (Substitution Theorem)**.** *Let f be a function and let u be a differentiable function of x. Assume $Range(u) \subset Dom(f)$, so that $f(u(x))$ is defined. Then*

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du.$$

One still has to guess at a good choice for the function u. You will get better with practice. Let's do a couple more examples.

Example 5. Find $\displaystyle\int \sqrt{2x+4}\,dx$.

Solution. Let $u = 2x + 4$. Then $du = 2\,dx$. Thus,

$$\int \sqrt{2x+4}\,dx = \frac{1}{2}\int 2\sqrt{2x+4}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}+C = \frac{1}{3}(2x+4)^{\frac{3}{2}}+C.$$

Note: we normally convert the result into terms of the original variable. □

Example 6. Find $\displaystyle\int \tan x\,dx$.

Solution. Rewrite the integrand as $\frac{\sin x}{\cos x}$. Let $u = \cos x$. Then

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C$$

Try using $u = \sin x$. It can be done this way, but it is quite a bit harder. □

Remark. Redo Examples 2 and 3 the short cut way. Redo Examples 4 and 5 the long way.

Example 7. Find $\displaystyle\int \frac{e^x + 1}{e^x}\,dx$.

Solution. It is a trick question! You do not need substitution. Just divide.

$$\int \frac{e^x + 1}{e^x}\,dx = \int 1 + e^{-x}\,dx = x - e^{-x} + C.$$

□

Finally we consider two examples with definite integrals. We will do each of them two different ways.

Example 8. Find $\displaystyle\int_0^2 x^2(x^3+1)^3\,dx$.

Solution 1. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$. Thus,

$$\int_0^2 x^2(x^3+1)^3\,dx = \frac{1}{3}\int_0^2 3x^2(x^3+1)^3\,dx = \frac{1}{3}\int_?^? u^3\,du.$$

But, when we switch to "du" what are the end points of the integral? Using "$\int_0^2$" is not correct. While x starts at 0 and ends at 2, u does not. Since $u = x^3 + 1$, the range of u is from 1 to 9. Thus,

$$\frac{1}{3}\int_{u(0)}^{u(2)} u^3\,du = \frac{1}{3}\int_1^9 u^3\,du = \frac{1}{3}\frac{1}{4}u^4\Big|_1^9 = \frac{1}{12}(9^4 - 1^4) = 546\tfrac{2}{3}.$$

□

Solution 2. However, we do not have to convert the end points. If we switch the result back in terms of x we can use the original end points.

$$\frac{1}{3}\int_?^? u^3\,du = \frac{1}{3}\frac{1}{4}u^4\Big|_?^? = \frac{1}{12}(x^3+1)^4\Big|_0^2 = \frac{1}{12}(9^4 - 1^4) = 546\tfrac{2}{3}$$

□

Example 9. Find $\int_0^\pi \sin^5 x \cos x \, dx$.

Solution 1. Let $u = \sin x$. (Using $u = \cos x$ also works, but is much harder.) Now, $du = \cos x \, dx$. We get,

$$\int_?^? u^5 \, du = \frac{1}{6} u^6 \Big|_?^? = \frac{1}{6} \sin^6 x \Big|_0^\pi = \frac{1}{6}(\sin^6 \pi - \sin^6 0) = \frac{1}{6}(0^6 - 0^6) = 0.$$

□

Now for the other way.

Solution 2. Since $u(0) = 0$ and $u(\pi) = 0$ we get

$$\int_0^0 u^5 \, du = 0.$$

Weird! For class discussion:
(1) What does $\int_a^a f(x) \, dx$ really mean?
(2) Show by using symmetry that $\int_0^\pi \sin^5 x \cos x \, dx = 0$. □

Example 10. Find $\int \sec x \, dx$.

Solution. This will require a dirty trick!

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx.$$

Notice that the drivative of the demoninator is the numerator! Let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x) \, dx$. Therefore

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\sec x + \tan x| + C.$$

□

Using the methods of this section - including in the problems - we can now integrate the six trigonometric functions. See the table below.

$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$	$\int \csc x\,dx = \ln\lvert\csc x - \cot x\rvert + C$

Elementary Problems.

1. $\displaystyle\int \cot y\,dy.$
2. $\displaystyle\int \frac{2q-7}{\sqrt{q+4}}\,dq.$
3. $\displaystyle\int \sec^2(6\theta+\pi)\,d\theta.$
4. $\displaystyle\int 5xe^{3x^2}\,dx.$
5. $\displaystyle\int \frac{1}{ax+b}\,dx.$
6. $\displaystyle\int \sin(\alpha B)\cos^2(\alpha B)\,dB.$
7. $\displaystyle\int \cos(4x+2)\,dx.$
8. $\displaystyle\int \tan 3u\,du.$
9. $\displaystyle\int \sec^2(3x)e^{\tan(3x)}\,dx.$
10. $\displaystyle\int \frac{x^2+x-2}{x+6}\,dx.$
11. $\displaystyle\int \frac{\rho^3+5}{2\rho}\,d\rho.$

12. $\displaystyle\int \frac{4x-6}{(x^2-3x+5)^5}\,dx.$

13. $\displaystyle\int (x+2)^7\,dx.$

14. $\displaystyle\int \cos\beta\sin\beta\,d\beta.$

15. $\displaystyle\int_0^1 (e^{2x}+2)^2\,dx.$

16. $\displaystyle\int_0^c 3(x-c)^6\,dx.$

17. $\displaystyle\int_{0.1}^{0.2} \frac{t^3}{\sec t^4}\,dt.$

18. $\displaystyle\int_{-3}^{3} \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx.$

19. $\displaystyle\int (6p^2+2)\sqrt[3]{p^3+p-1}\,dp.$

20. $\displaystyle\int \frac{0.7H_\theta-2.1}{7H_\theta}\,dH_\theta.$

21. $\displaystyle\int \frac{\sqrt{\cos^2 3\zeta+\sin^2 3\zeta}}{17.3}\,d\zeta.$

Sophisticated Problems.

22. $\displaystyle\int \frac{ax+b}{cx+d}\,dx$

23. $\displaystyle\int \phi\sec 3\phi^2\tan 3\phi^2\,d\phi$

24. $\displaystyle\int \frac{2x}{1+4x^2}\,dx$

25. $\displaystyle\int \sin^3 x^2\,dy$. Hint: this is a trick question.

26. $\int \frac{1}{\sqrt{t}(1-\sqrt{t})}\,dt$

27. $\int \sin^2 s\,ds$. Hint: use the identity $\sin^2 x = \frac{1-\cos 2x}{2}$.

28. $\int \cos^4 3\alpha\,d\alpha$.

29. $\int \tan^4 x\,dx$.

30. $\int \sin 3x \cos 2x\,dx$.

31. $\int \sin^5 x\,dx$.

32. $\int \sec x\,dx$. Hint: multiply by $\frac{\sec x + \tan x}{\sec x + \tan x}$.

33. $\int \csc x\,dx$.

34. $\int \frac{y^3 - 2y^2 + y - 4}{y-3}\,dy$.

35. $\int \frac{1}{1+\cos 5\theta}\,d\theta$.

36. $\int \cot^4 2x \csc^2 2x\,dx$.

37. $\int \cos t \sin 2t\,dt$.

38. $\int \frac{e^{2x}-1}{e^{2x}+3}\,dx$.

39. $\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$.

40. $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx.$
Hint: Complete the square, $x^2 - 2x = (x-1)^2 - 1$, use $u = x - 1$ and recall the derivative formulas for the inverse trig functions.

41. $\displaystyle\int \frac{w+6}{w^2+1}\,dw.$

42. $\displaystyle\int \frac{w^3-2w^2+4}{w^2+1}\,dw.$

43. $\displaystyle\int \frac{\sin 8\theta}{9+\sin^4\theta}\,d\theta.$

44. $\displaystyle\int \frac{1}{(x+1)(x-2)}\,dx.$ Hint: Show $\frac{1}{(x+1)(x-2)} = \frac{1}{3}\left[\frac{1}{x-2} - \frac{1}{x+1}\right].$

45. $\displaystyle\int \frac{1}{x^2-9}\,dx.$

46. $\displaystyle\int \frac{x}{x^2-9}\,dx.$

47. $\displaystyle\int \frac{\cos^2(\tan x)}{\cos^2 x}\,dx.$

48. $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx.$

49. $\displaystyle\int \cot\rho\sec^2\rho\,d\rho.$