

Example of Nonuniform Convergence

Example. Let $g_n(x) = \frac{nx + \sin^2 x}{3n + \cos^2 x}$. We will show that $\lim_{n \rightarrow \infty} g_n(x) = x/3$ pointwise, but not uniformly. My intuition was that the convergence would be uniform. When I cooked up this sequence of functions I was trying to find an example of uniform convergence. I tried to prove that the convergence was uniform and for a while thought I had done so!

Proof. Rewrite $g_n(x)$ as

$$g_n(x) = \frac{x + \frac{\sin^2 x}{n}}{3 + \frac{\cos^2 x}{n}}.$$

Thus, by the rules for limits, we have for any value x that

$$\lim_{n \rightarrow \infty} g_n(x) = \frac{x + 0}{3 + 0} = \frac{x}{3}.$$

This establishes pointwise convergence.

Let $\epsilon = 1$. Choose any $n \in \mathbb{N}$. Let $x = k\pi$ where $k \in \mathbb{N}$. Then

$$\left| g_n(x) - \frac{x}{3} \right| = \left| \frac{nk\pi}{3n+1} - \frac{k\pi}{3} \right| = \frac{k\pi}{3(3n+1)}.$$

By the Archimedean Property (page 25) $\exists k \in \mathbb{N}$ s.t.

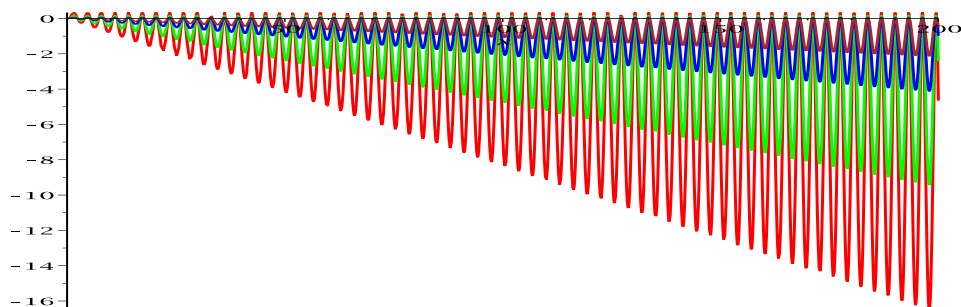
$$\frac{k\pi}{3(3n+1)} > 1.$$

Hence you cannot find one $n \in \mathbb{N}$ that gives

$$|g_n(x) - x/3| < 1 \quad \forall x \in \mathbb{R}.$$

□

Below I have graphed $y = g_n(x) - x/3$ for $n = 1$ (red), 2(green), 5(blue) and 10(brown) over $x \in [0, 250]$. We will discuss this in class.



Conclusion. What does it mean to be a well educated person? It means, at least in part, the willingness to give up a belief when the evidence undermines it. Of course outside of mathematics evidence is not always black and white. One has to consider the preponderance of the evidence. One has to navigate evidence that may be ambiguous. Sometimes your mind plays tricks on you. The social networks one is enmeshed in will influence what sources of evidence one finds trustworthy. Often times we have to make decisions while still uncertain. A well educated person is aware of all this and so can adapt and change.

However, there are traps that highly educated people tend to fall into. One is being arrogant or being intolerant of people who are not so highly educated. There are plenty people who never went to college who are in fact open minded, creative and can reason logically. Being well educated and highly educated are not the same thing.