Taylor's Theorem

Def: Here and below let \( I \) be an open interval. It may be
bond or unbounded. Let \( f: I \to \mathbb{R} \). Let \( c \in I \).
\( f^{(k)}(c) \) means the \( k \)-th derivative at \( c \); \( f^{(0)} = f \).

Suppose \( f^{(k)}(c) \) exists for \( 0 \leq k \leq n \). Then the
\( n \)-th order Taylor polynomial of \( f \) at \( c \) is

\[
P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k
\]

If \( f^{(k)}(c) \) exists for all \( k \geq 0 \) then the Taylor
Series of \( f \) at \( c \) is

\[
T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k.
\]

Note: The radius of convergence could be zero, or infinity
or any other value in \((0, \infty)\). Even if it converges, it
might not converge to \( f(x) \).

For \( n \geq 1 \) we define the remainder to be

\[
R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x-c)^k.
\]

Clearly, \( T(x) = f(x) \) if \( f \) \( \lim_{n \to \infty} R_n(x) = 0 \).
31.3 (Taylor's Theorem, Version 1)

Let \( f: I \to \mathbb{R} \). Suppose \( f^{(n)}(x) \) exists \( \forall x \in I \).

Then for all \( x \in I - \{ c \} \) (i.e., \( x \neq c \)), \( \exists y \) between \( c \) and \( x \) s.t.

\[
R_n(x) = \frac{f^{(n)}(y)}{n!} (x-c)^n.
\]

**Proof**

Fix an \( x \neq c \) in \( I \) and an \( n \geq 1 \). Then \( \exists! M \in \mathbb{R} \) s.t.

\[
f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x-c)^k + \frac{M (x-c)^n}{n!}.
\]

We will find a value \( y \) in between \( c \) and \( x \) s.t.

\[
f^{(n)}(y) = M.
\]

Let \( g(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x-c)^k + \frac{M (x-c)^n}{n!} - f(x) \).

**Claim:** \( g(c) = 0 \). **Proof:** \( g(c) = f(c) + \sum_{k=1}^{n-1} 0 + 0 - f(c) = 0 \).

**Claim:** \( g^{(k)}(c) = 0 \) for \( k = 1, 2, \ldots, n-1 \).

**Proof**

\[
g'(t) = \sum_{k=1}^{n-1} \frac{f^{(k)}(c)}{(k-1)!} (t-c)^{k-1} + \frac{M (t-c)^{n-1}}{(n-1)!} - f'(t).
\]

\[
g'(c) = f''(c) - f'(c) = 0.
\]

\[
g''(t) = \sum_{k=2}^{n-1} \frac{f^{(k)}(c)}{(k-2)!} (t-c)^{k-2} + \frac{M (t-c)^{n-2}}{(n-2)!} - f''(t)
\]

\[
g''(c) = f''(c) - f''(c) = 0, \quad \exists \epsilon \in (c).
\]
From the definition of \( g \) we know \( g(x) = 0 \).

Thus \( \exists x_1 \) between \( c \) and \( x \) s.t. \( g'(x_1) = 0 \). (Rolle's Thm)

\[ g(x) \begin{array}{c} \nearrow \end{array} \]

\[ c \quad x_1 \quad x \]

\( \exists x_2 \) between \( c \) and \( x_1 \) s.t. \( g''(x_2) = 0 \). (R's Thm)

(Since \( g'(c) = g'(x_1) = 0 \).)

\[ g''(x_2) = 0 \quad g'(x_1) = 0 \]

\[ c \quad x_2 \quad x_1 \quad x \]

\( \exists x_3 \) between \( c \) and \( x_2 \) s.t. \( g'''(x_3) = 0 \).

Continue until we find \( x_n \) between \( c \) and \( x_{n-1} \) s.t. \( g^{(n)}(x_n) = 0 \).

Let \( y = x_n \). Now, remember what we are trying to prove, \( f^n(y) = M \). Now

\( g^{(n)}(t) = 0 + M - f^{(n)}(t) \), for all \( t \in I \).

Let \( t = y \). Then \( g^n(y) = 0 = M - f^{(n)}(y) \Rightarrow M = f^{(n)}(y) \).

Now \( R^n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n \). □
3.4 Corollary Let $f : I \to \mathbb{R}$ be infinitely differentiable on $I$. Assume all the derivatives are bounded by a constant $C$. That is

$$|f^{(n)}(x)| \leq C \quad \forall x \in I, \ n = 0, 1, 2, 3, \ldots$$

Then

$$\lim_{n \to \infty} R_n(x) = 0 \quad \forall x \in I.$$ 

**Proof**

Let $x \in I$. By the last theorem

$$|R_n(x)| \leq \frac{C}{n!} |x-c|^n \quad \forall n \in \mathbb{N}.$$ 

The limit $\lim_{n \to \infty} \frac{|x-c|^n}{n!} = 0$ since by the Ratio Test

the series $\sum_{n=0}^{\infty} \frac{|x-c|^n}{n!}$ converges, so the

$n$-th term test requires $\lim_{n \to \infty} \frac{|x-c|^n}{n!} = 0$. \qed
Ex 3: This is an example of a function that has all derivatives existing on $\mathbb{R}$, but whose Taylor series fails to converge to it. Functions that have their Taylor series are called analytic functions. This example shows that while all analytic functions are infinitely differentiable, the reverse is false.

The notation is $C^\omega(I) =$ all infinitely diff. func's on $I$, $C^\omega(I) =$ all analytic func's on $I$.

$C^\omega(I) \subset C^\infty(I)$, $C^\omega(I) \neq C^\infty(I)$.

Let $f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0, \\ 0 & x \leq 0. \end{cases}$

It is clear $f^{(n)}(x)$ exists for $x = 0$. We will show $f^{(n)}(0) = 0$ for all $n$. Hence, the Taylor Series at $x = 0$ has all terms $= 0$. Thus, it converges to the zero function $f(x)$ on $\mathbb{R}$. So, it cannot converge to $f(x)$ on any open interval containing 0.
Claim: \( f^{(n)}(x) = \begin{cases} e^{-\frac{1}{x}} p_n(x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \)

where \( p_n(x) \) is a polynomial.

**Proof** This is obvious for \( x < 0 \). For \( x > 0 \) will use induction. The case \( n = 0 \) is immediate. Suppose

\[
f^{(n)}(x) = e^{-\frac{1}{x}} p(x) \quad \text{for some poly. } p(x).
\]

\[
f^{(n+1)}(x) = (e^{-\frac{1}{x}})' p(x) + e^{-\frac{1}{x}} [p(x)]'
\]

\[
= e^{-\frac{1}{x}} \left( \frac{1}{x^2} p(x) + e^{-\frac{1}{x}} p'(x) \left( \frac{1}{x^2} \right) \right)
\]

\[
= e^{-\frac{1}{x}} \left( \frac{1}{x^2} \right) (p(x) + p'(x)).
\]

Since \( \frac{1}{x^2} (p(x) + p'(x)) \) is a poly in \( \frac{1}{x} \), we are done.

Now we have to check \( x = 0 \). We will use the following fact that you can prove using L'Hopital's Rule:

\[
\lim_{y \to \infty} y^k e^{-y} = 0 \quad \forall k \in \mathbb{N}.
\]
Let \( n = 1 \), \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1}{x} f(x) \).

The limit as \( x \to 0^- \) is zero since \( f(x) = 0 \) for \( x < 0 \).

The limit as \( x \to 0^+ \) is \( \lim_{x \to 0^+} \frac{1}{x} e^{-\frac{x}{2}} = \lim_{x \to \infty} y e^{-y} = 0 \).

Thus \( f'(0) = 0 \).

Assume \( f^{(n)}(0) = 0 \). Then

\[
\frac{d^n f}{dx^n}(0) = \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0} \frac{1}{x} f^{(n)}(x).
\]

Again the limit as \( x \to 0^- \) is zero.

\[
\lim_{x \to 0^+} \frac{1}{x} f^{(n)}(x) = \lim_{x \to 0^+} \frac{1}{x} e^{-\frac{x}{2}} p_n(x) = \lim_{y \to \infty} y p_n(y) e^{-y} = 0.
\]