§32. Continued.
Assume \( f \) is a bounded function on \([a, b]\).

Lemma 32.2. Let \( P \subset Q \) be partitions of \([a, b]\). (\( Q \) is called a refinement of \( P \).) Then

\[
L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).
\]

**Proof.** The middle inequality we already discussed.

Consider the refinement. Suppose \( Q \) has one more point than \( P \). Then, if \( P = t_0, t_1, \ldots, t_n \), let

\[ Q = \{t_0, \ldots, t_{k-1}, u, t_k, \ldots, t_n\}. \]

\[
L(f, P) = \sum_{i=1}^{n} m(f, [t_{i-1}, t_i])(t_i - t_{i-1})
\]

\[
L(f, Q) = \sum_{i=1}^{k-1} m(f, [t_{i-1}, t_i])(t_i - t_{i-1}) + m(f, [t_{k-1}, u])(u - t_{k-1})
\]

\[ + m(f, [u, t_k])(t_k - t_{k-1}) + \sum_{i=k+1}^{n} m(f, [t_{i-1}, t_i])(t_i - t_{i-1}) \]

Thus,

\[
L(f, Q) - L(f, P) = m(f, [t_{k-1}, u])(u - t_{k-1})
\]

\[ + m(f, [u, t_k])(t_k - u) - m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \]  

Now, \( m(f, [t_{k-1}, t_k]) \leq m(f, [t_{k-1}, u]) \) and \( m(f, [t_{k-1}, t_k]) \leq m(f, [u, t_k]) \).
Thus,
\[ m(f, [t_{k-1}, t_k]) (u - t_{k-1}) \leq m(f, [u, t_k]) (u - t_{k-1}) \]
and
\[ m(f, [t_{k-1}, t_k]) (t_k - u) \leq m(f, [u, t_k]) (t_k - u) \]
Add these to get
\[ m(f, [t_{k-1}, t_k]) (t_k - t_{k-1}) \leq m(f, [t_{k-1}, u]) (u - t_{k-1}) \]
\[ + m(f, [u, t_k]) (t_k - u) \]
Thus from (a) we have
\[ L(f, Q) - L(f, P) \geq 0 \]
or
\[ L(f, P) \leq L(f, Q) \]

Now suppose \( Q \) has \( m \geq 1 \) more members than \( P \).
Call them \( q_1, q_2, \ldots, q_m \) in any order.

Let \( Q_1 = Q \cup \{q_1, q_2, \ldots, q_m\} \)
Then
\[ L(f, Q) \leq L(f, Q_1) \leq L(f, Q_2) \leq \ldots \leq L(f, Q) \]
The proof for \( U(f, Q) \leq U(f, P) \) is similar.

**Lemma 3.2.3** Let \( P \) and \( R \) be partitions of \([a, b] \). Then
\[ L(f, P) = U(f, R) \]

\( \text{pf} \)
Let \( Q = P \cup R \). Then
\[ L(f, P) \leq L(f, Q) \leq P(f, Q) \leq P(f, R) \]
Thm 32.4  \[ L(f) \leq U(f). \]

*Proof*  Recall \[ U(f) = \inf \{ U(f, Q) \mid Q \text{ is a partition of } [a,b] \} \]

Let \( P \) be any partition of \( [a,b] \) then since

\[ L(f, P) \leq U(f, Q) \quad \forall \ Q, \text{ partitions of } [a,b] \]

\[ L(f, P) \leq \inf \{ U(f, Q) \mid \text{ all partitions } Q \} = U(f). \]

Recall \[ L(f) = \sup \{ L(f, R) \mid \text{ all partitions } R \text{ of } [a,b] \}. \]

For any \( R \), \[ L(f, R) \leq U(f). \]

Thus \[ \sup \{ L(f, R) \mid \ldots \} \leq U(f). \]

That is \[ L(f) \leq U(f). \]
**Theorem 32.5 (The Cauchy criterion for integrability).** A bounded function $f$ on $[a, b]$ is Darboux integrable if and only if

$$\forall \varepsilon > 0 \; \exists \; \text{a partition } P \text{ of } [a, b], \; \delta > 0.$$

$$U(f, P) - L(f, P) < \varepsilon.$$

**Proof.** First suppose $f$ is integrable. Let $\varepsilon > 0$.

$\exists$ a partition $P_1$ such that $L(f, P_1) > L(f) - \frac{\varepsilon}{2}.$

$\exists$ a partition $P_2$ such that $U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$

Let $P = P_1 \cup P_2$. By Lemma 32.2,

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

Thus,

$$U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_1) < U(f) + \frac{\varepsilon}{2} - \left[ L(f) - \frac{\varepsilon}{2} \right] = U(f) - L(f) + \varepsilon.$$

But since $f$ is Darboux integrable, $U(f) = L(f)$. Thus,

$$U(f, P) - L(f, P) < \varepsilon.$$
Now we have to do the other direction since this is a "if and only if" theorem.

Suppose \( \forall \varepsilon > 0 \), \( \exists \) a partition \( P \) of \([a, b] \) s.t.

\[
U(f, P) - L(f, P) < \varepsilon.
\]

We know that \( L(f) \leq U(f) \). So, if we prove \( U(f) \leq L(f) \) we know they are equal. Here we see...

\[
U(f) = U(f, P) = U(f, P) - L(f, P) + L(f, P) < \varepsilon + L(f, P)
\]

Thus \( U(f) - L(f) \leq \varepsilon, \forall \varepsilon > 0 \).

Thus \( U(f) - L(f) \leq 0 \), or \( U(f) \leq L(f) \).

Hence \( U(f) = L(f) \), thus \( f \) is Darboux integrable by definition.