§32 continued.

Assume f is a bdd function on $[a,b]$.

<u>Lemma 32.2</u> Let $P \subset Q$ be partitions of $[a,b]$. (Q is called a <u>refinement</u> of P.) Then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

<u>Pf</u> The middle inequality we already discussed. ~~We will use induction~~. Suppose Q has one more pt than P. If $P = \{t_0, t_1, \dots, t_n\}$, let

$$Q = \{t_0, \dots t_{k-1}, u, t_k, \dots, t_n\}.$$

$$L(f,P) = \sum_{i=1}^{n} m(f,[t_{i-1},t_i]) \cdot (t_i - t_{i-1})$$

$$L(f,Q) = \sum_{i=1}^{k-1} m(f,[t_{i-1},t_i])(t_i - t_{i-1}) + m(f,[t_{k-1},u])(u - t_{k-1})$$

$$+ m(f,[u,t_k])(t_k - u) + \sum_{i=k+1}^{n} m(f,[t_{i-1},t_i])(t_i - t_{i-1})$$

Thus, $L(f,Q) - L(f,P) = m(f,[t_{k-1},u])(u - t_{k-1})$

$$+ m(f,[u,t_k])(t_k - u) - m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \quad (*)$$

Now $m(f,[t_{k-1},t_k]) \le m(f,[t_{k-1},u])$ and
$m(f,[t_{k-1},t_k]) \le m(f,[u,t_k])$

Thus, $\quad m(f,[t_{k-1},t_k])(u-t_{k-1}) \le m(f,[t_{k-1},u])(u-t_{k-1})$

and $\quad m(f,[t_{k-1},t_k])(t_k-u) \le m(f,[u,t_k])(t_k-u)$

Add these to get

$$m(f,[t_{k-1},t_k])(t_k-t_k) \le m(f,[t_{k-1},u])(u-t_{k-1}) + m(f,[u,t_k])(t_k-u)$$

Thus from (*) we have $L(f,Q)-L(f,P) \ge 0$ or

$$L(f,P) \le L(f,Q).$$

Now suppose Q has $m \ge 1$ more members than P. Call them $q_1, q_2 \dots q_m$, in any order.

Let $\quad Q_1 = P \cup \{q_1\},\ Q_2 = Q_1 \cup \{q_2\}, \dots Q_m = Q_{m-1} \cup \{q_m\} = Q.$

Then $\quad L(f,P) \le L(f,Q_1) \le L(f,Q_2) \le \cdots \le L(f,Q).$

The proof for $U(f,Q) \le U(f,P)$ is similar. ■

<u>Lemma 3.2.3</u> Let P and R be partitions of $[a,b]$. Then

$$L(f,P) \le U(f,R).$$

<u>Pf</u> Let $Q = P \cup R$. Then

$$L(f,P) \le L(f,Q) \le P(f,Q) \le P(f,R)$$ ■

Thm 32.4 $L(f) \le U(f)$.

Pf Recall $U(f) = \inf \{ U(f, Q) \mid Q \text{ is a partition of } [a,b] \}$

Let P be any partition of $[a,b]$ then since

$$L(f, P) \le U(f, Q) \quad \forall_{Q, \text{ partitions of } [a,b]}$$

$$L(f, P) \le \inf \{ U(f, Q) \mid \text{all partitions} \} = U(f).$$

Recall $L(f) = \overset{\sup}{\{} L(f, R) \mid \text{all partitions } R \text{ of } [a,b] \}$.

For any R, $L(f, R) \le U(f)$.

Thus $\sup \{ L(f, R) \mid \cdots \} \le U(f)$.

That is $L(f) \le U(f)$.

Thm 32.5 (The Cauchy criterion for integrability). A bdd function f on $[a,b]$ is Darboux integrable iff

$$\forall \varepsilon > 0 \ \exists \text{ a partition } P \text{ of } [a,b], \text{ s.t.}$$

$$U(f,P) - L(f,P) < \varepsilon.$$

Pf First suppose f is integrable. Let $\varepsilon > 0$.

$\exists$ a partition P_1 s.t. $L(f,P_1) > L(f) - \frac{\varepsilon}{2}$.

$\exists$ a partition P_2 s.t. $U(f,P_2) < U(f) + \frac{\varepsilon}{2}$.

Let $P = P_1 \cup P_2$. By Lemma 32.2

$$L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2).$$

Thus,
$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,P_2) - L(f,P_1) \\ &< U(f) + \frac{\varepsilon}{2} - \left[L(f) - \frac{\varepsilon}{2}\right] \\ &= U(f) - L(f) + \varepsilon. \end{aligned}$$

But since f is Darboux integrable, $U(f) = L(f)$.
Thus,
$$U(f,P) - L(f,P) < \varepsilon.$$

Now we have to do the other direction since this is a "if and only if" theorem.

Suppose $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$ s.t.

$$U(f,P) - L(f,P) < \varepsilon.$$

We know that $L(f) \le U(f)$. So, if we prove $U(f) \le L(f)$ we know they are equal. Here we go...

$$U(f) \le U(f,P) = U(f,P) - L(f,P) + L(f,P) < \varepsilon + L(f,P)$$
$$\le \varepsilon + L(f)$$

Thus $U(f) - L(f) \le \varepsilon$, $\forall \varepsilon > 0$.

Thus $U(f) - L(f) \le 0$, or $U(f) \le L(f)$.

Hence $U(f) = L(f)$. Thus f is Darboux integrable by definition. $\boxed{}$