Absolute Values and the Triangle Inequality

Definition. For any real number a we define the **absolute value** of a as

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

Useful Fact. For all real numbers $-|a| \leq a \leq |a|$.

Proof. I'll need to use that $-0 = 0$ so I'll prove that first.

$$\begin{array}{rcll} (i) & & 0 + (-0) = 0 & \text{by A4.} \\ (ii) & & 0 = 0 + 0 & \text{by A3.} \\ (i)\&(ii) & \Longrightarrow & 0 + (-0) = 0 + 0. & \\ \text{Thus,} & & -0 = 0 & \text{by Thm 3.1(i).} \end{array}$$

Assume $0 \leq a$. Then $a \leq |a|$ since $|a| = a$. By order axiom O3, $0 \leq |a|$. Thus, by Theorem 3.2(i) $-|a| \leq -0 = 0$. Again by O3, $-|a| \leq a$. Thus, we have

$$-|a| \leq a \leq |a|.$$

Now assume $a < 0$. Then $|a| = -a$. As before $0 = -0 < -a + |a|$. Thus, $a \leq |a|$. Now using field axioms A2 and A3 we see that $|a| = -a$ implies $a = -|a|$. Thus,

$$-|a| \leq a \leq |a|.$$

□

Another Handy Fact. If $-y \leq x \leq y$, then $|x| \leq y$.

Proof. The first inequality is equivalent to $-x \leq y$. Since $|x|$ equals x or $-x$, the result follows. □

Theorem. The Triangle Inequality (3.5(iii) in your textbook). For all real numbers a and b we have

$$|a+b| \leq |a| + |b|.$$

Long Proof. I'll use a two column format.

$$\begin{array}{rrcrl} (i) & -|a| \leq a & \Longrightarrow & -|a| - |b| \leq a - |b| & \text{by O4.} \\ (ii) & -|b| \leq b & \Longrightarrow & a - |b| \leq a + b & \text{by O4.} \\ (iii) & a \leq |a| & \Longrightarrow & a + b \leq |a| + b & \text{by O4.} \\ (iv) & b \leq |b| & \Longrightarrow & |a| + b \leq |a| + |b| & \text{by O4.} \\ (v) & (i)\&(ii) & \Longrightarrow & -|a| - |b| \leq a + b & \text{by O3.} \\ (vi) & (iii)\&(iv) & \Longrightarrow & a + b \leq |a| + |b| & \text{by O3.} \end{array}$$

By a homework problem $-x = -1 \cdot x$. Thus,

$$-|a| - |b| = -1 \cdot |a| + -1 \cdot |b| = -1 \cdot (|a| + |b|).$$

Then (v) implies $-(a+b) \leq |a| + |b|$ by Theorem 3.2(ii).

Now we have

$$a + b \leq |a| + |b| \quad \text{and} \quad -(a+b) \leq |a| + |b|.$$

By definition $|a+b|$ is equal to $a+b$ or $-(a+b)$. Either way we have

$$|a+b| \leq |a| + |b|.$$

□

Short Proof. Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$ we know

$$-|a| - |b| \leq a + b \leq |a| + |b|.$$

Thus,

$$|a+b| \leq |a| + |b|.$$

□

Which do you like better? Which do you believe?

Importance of the Triangle Inequality

The Triangle Inequality has many applications and generalizations. We will use the Triangle Inequality many times in this course. We mention a few generalizations here. By induction one can show

$$\Big|\sum_{i=1}^{n} a_i\Big| \leq \sum_{i=1}^{n} |a_i|.$$

This works also if the a_i's are vectors or complex numbers where the $|\cdot|$ means magnitude.

Using the theory of limits it can be shown that

$$\Big|\sum_{i=1}^{\infty} a_i\Big| \leq \sum_{i=1}^{\infty} |a_i|,$$

when both sums converge. Again the a_i's can be real, complex or vectors.

Using calculus it can be shown that

$$\Big|\int_a^b f(x)\,dx\Big| \leq \int_a^b |f(x)|\,dx,$$

when both integrals exist.

Another generalization, that is Exercise 3.5(b) in your textbook, is

$$\big||a| - |b|\big| \leq |a - b|$$

for all real numbers a and b. It also holds when a and b are complex numbers or vectors.