

## Completeness Axiom

**Definitions.** Let  $S$  be a nonempty subset of  $\mathbb{R}$ . An **upper bound** for  $S$  is any real number  $b$  such that if  $x \in S$  then  $x \leq b$ . A **least upper bound (lub)** for  $S$  is any real number  $c$  such that  $c$  is an upper bound for  $S$  and if  $b$  is another upper bound of  $S$  then  $c < b$ . If  $S$  contains its least upper bound  $c$  then we say  $c$  is the **maximum** of  $S$ .

**Examples.** Let  $S = (-1, 1)$ . Then 17 is an upper bound of  $S$  and 1 is the least upper bound of  $S$ . In the case  $S$  does not have a maximum. Let  $T = (0, 1) \cup \{2\}$ . Then 2 is the least upper bound and the maximum of  $T$ .

The terms **lower bound**, **greatest lower bound (glb)** and **minimum** are defined similarly.

**Examples.** Let  $A = (0, 3) \cup \mathbb{N}$ . Then  $A$  does not have an upper bound. Any negative number is a lower bound and 0 is the greatest lower bound. The set  $A$  does not have a minimum.

Let  $B = \left\{ \frac{n+1}{n} \mid n \in \mathbb{N} \right\}$ . Then 2 is the maximum and 1 is the greatest lower bound. The set  $B$  does not have a minimum.

**The Completeness Axiom for  $\mathbb{R}$ .** If  $S$  is a nonempty subset of  $\mathbb{R}$  that is bounded from above then  $S$  has a least upper bound.

In MATH 352 the Completeness Axiom is assumed to be true for  $\mathbb{R}$ . In MATH 452 we prove the Completeness Axiom is true for  $\mathbb{R}$ .

The Completeness Axiom is **false** for  $\mathbb{Q}$ . Let  $A = \{r \in \mathbb{Q} \mid r^2 < 2\}$ . Then  $A$  is not empty since  $1 \in A$ . You can show that 3 is an upper bound for  $A$ . But  $A$  does not have a least upper bound in  $\mathbb{Q}$ . If it did, say  $r_*$  is the lub, then it can be shown that  $(r_*)^2 = 2$ , but we know there is no such rational number. If we regard  $A$  as a subset of  $\mathbb{R}$  then it does have a lub that is called  $\sqrt{2}$ . In MATH 452 we prove that  $(\sqrt{2})^2 = 2$ . In MATH 352 we assume this.

**Corollary.** Using the Completeness Axiom it is easy to prove that if  $S$  is a nonempty subset of  $\mathbb{R}$  that is bounded from below then  $S$  has a greatest lower bound. See textbook for proof.

**Definitions.** Let  $S \subset \mathbb{R}$ . Then the **supremum** and **infimum** of  $S$  are defined as follows.

$$\sup S = \begin{cases} +\infty & \text{if } S \neq \emptyset \text{ and is not bounded above,} \\ \text{lub } S & \text{if } S \neq \emptyset \text{ and is bounded above,} \\ -\infty & \text{if } S = \emptyset. \end{cases}$$

$$\inf S = \begin{cases} -\infty & \text{if } S \neq \emptyset \text{ and is not bounded below,} \\ \text{glb } S & \text{if } S \neq \emptyset \text{ and is bounded below,} \\ +\infty & \text{if } S = \emptyset. \end{cases}$$

**Theorem.** The Archimedean Property. Let  $a$  and  $b$  be positive real numbers. Then  $\exists n \in \mathbb{N}$  such that  $na > b$ .

The proof uses the Completeness Axiom and is harder than you would think!

*Proof.* Suppose not. Then  $\exists a > 0$  and  $b > 0$  such that  $\forall n \in \mathbb{N} na \leq b$ .  
Let

$$S = \{na \mid n \in \mathbb{N}\}.$$

Since  $b$  is an upper bound for  $S$ ,  $S$  must have a lub. Call it  $s_0$ .

Since  $0 < a$  we have  $s_0 < s_0 + a$  and hence  $s_0 - a < s_0$ .

$\exists n_0 \in \mathbb{N}$  such that  $s_0 - a < n_0 a$  because  $s_0 - a$  is less than the least upper bound of  $S$ .

Hence,  $s_0 < (n_0 + 1)a$ .

But this means  $(n_0 + 1)a \notin S$ . Hence, our supposition was foolish!  
The Archimedean Property has been vindicated!!  $\square$

**Two Corollaries.** These will be useful in proofs.

1. If  $a > 0$ , then  $\exists n \in \mathbb{N}$  s.t.  $\frac{1}{n} < a$ .
2. If  $b > 0$ , then  $\exists n \in \mathbb{N}$  s.t.  $n < b$ .

*Proofs.*

1. Let  $a > 0$  and  $b = 1.0$ . By the Archimedean Property  $\exists n \in \mathbb{N}$  s.t.  $na > 1$ . Then  $\frac{1}{n} < a$ .
2. Let  $a = 1 > 0$  and  $b > 0$ . By the Archimedean Property  $\exists n \in \mathbb{N}$  s.t.  $n \cdot 1 > b$ . Then  $n > b$ .

**Denseness of  $\mathbb{Q}$  in  $\mathbb{R}$ .**  $\forall a$  and  $b$  in  $\mathbb{R}$ , with  $a < b$ ,  $\exists \frac{m}{n} \in \mathbb{Q}$  s.t.

$$a < \frac{m}{n} < b.$$

*Proof.* Since  $b - a > 0$ ,  $\exists n \in \mathbb{N}$  s.t.  $n(b - a) > 1$  (\*).

$\exists k \in \mathbb{N}$  s.t.  $-k < na < nb < k$ . (Why?)

Let  $m$  be the smallest number in  $\{-k, -k + 1, \dots, k - 1, k\}$  that is bigger than  $na$ . Then

$$-k < na < m \text{ and } m - 1 \leq na.$$

Thus, using (\*),

$$na < m \leq na + 1 < nb.$$

Thus,

$$a < \frac{m}{n} < b.$$

□