Ordered Fields Work Sheet

Definition. Let S be a set. A **binary operation** is a map from $S \times S$ to S.

Definition. An **ordered field** is a set $\mathbb{F}$ with more than one member together with two binary operations, addition $+$ and multiplication $\cdot$ and an order relation $\leq$ satisfying the axioms below for all a, b, and c in $\mathbb{F}$. (The $\cdot$ is often left unwritten.)

A1. $a + (b + c) = (a + b) + c$.
A2. $a + b = b + a$.
A3. $\exists 0 \in \mathbb{F}$ such that $a + 0 = a$.
A4. $\exists -a \in \mathbb{F}$ such that $a + (-a) = 0$.
M1. $a(bc) = (ab)c$.
M2. $ab = ba$.
M3. $a \cdot 1 = a$.
M4. If $a \neq 0$, $\exists a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$.
DL. $a(b + c) = ab + ac$.
O1. Either $a \leq b$ or $b \leq a$ is true.
O2. If $a \leq b$ and $b \leq a$, then $a = b$.
O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
O4. If $a \leq b$, then $a + c \leq b + c$.
O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Note: Given $\leq$ then $\geq$, $<$ and $>$ are defined as usual.

From these beginnings the following can be proven.

Theorem 3.1 Let $\mathbb{F}$ be a field. Then $\forall$ a, b, c in $\mathbb{F}$ the following hold.

(i) If $a + c = b + c$, then $a = b$.
(ii) $a \cdot 0 = 0$.
(iii) $(-a)b = -(ab)$.
(iv) $(-a)(-b) = ab$.
(v) If $ac = bc$ and $c \neq 0$, then $a = b$.
(vi) If $ab = 0$ then either $a = 0$ or $b = 0$.

Theorem 3.2 Let $\mathbb{F}$ be an ordered field. Then $\forall$ a, b, c in $\mathbb{F}$ the following hold.

(i) If $a \leq b$, then $-b \leq -a$.
(ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.
(iii) If $0 \leq a$ and $0 \leq b$, then $bc \leq ac$.
(iv) $0 \leq a^2$.
(v) $0 < 1$.
(vi) If $0 < a$, then $0 < a^{-1}$.
(vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Some Proofs. You fill in the missing steps or reasons.
3.1(i) If $a + c = b + c$, then $a = b$.

$$\begin{array}{rcll} a+c &=& b+c & \text{is given.} \\ (a+c)+(-c) &=& (b+c)+(-c) & \text{by def. of binary op.} \\ a+(c+(-c)) &=& b+(c+(-c)) & \text{by axiom A1 applied to both sides.} \\ a+0 &=& b+0 & \text{by \_\_\_.} \\ a &=& b & \text{by \_\_\_.} \end{array}$$

3.1(ii) $a \cdot 0 = 0$.

$$\begin{array}{rcll} a\cdot 0 &=& a(0+0) & \text{by A3.} \\ a\cdot 0 &=& a\cdot 0 + a\cdot 0 & \text{by \_\_\_.} \\ a\cdot 0 + 0 &=& a\cdot 0 + a\cdot 0 & \text{by \_\_\_.} \\ 0 + a\cdot 0 &=& a\cdot 0 + a\cdot 0 & \text{by A2.} \\ 0 &=& a\cdot 0 & \text{by 3.1(i).} \end{array}$$

3.1(iii) $(-a)b = -(ab)$.

$$\begin{array}{rcll} 0\cdot b &=& 0 & \text{by 3.1(ii).} \\ (a+(-a))b &=& 0 & \text{by \_\_\_.} \\ ab+(-a)b &=& 0 & \text{by DL.} \\ ab+-(ab) &=& 0 & \text{by A4.} \\ ab+-(ab) &=& ab+(-a)b & \text{since 0=0.} \\ (-a)b &=& -(ab) & \text{by 3.1\_\_.} \end{array}$$

3.1(iv) $(-a)(-b) = ab$.

It will be useful to first prove that $-(-x) = x$. Since $(-x) + x = 0$ and $(-x) + (-(-x)) = 0$, Theorem 3.1(i) says that $-(-x) = x$.

$$\begin{array}{rcll} (-a)(-b) &=& -(a(-b)) & \text{3.1(iii)} \\ &=& -((-b)a) & \text{M2} \\ &=& -(-(ba)) & \text{3.1(iii)} \\ &=& ab & \end{array}$$

3.1(v) $ac = bc \;\&\; c \neq 0 \implies a = b$.

First, c^{-1} exists by ___.

$$\begin{array}{rcll} (ac)c^{-1} & = & (bc)c^{-1} & \text{by def. of binary op.} \\ a(cc^{-1}) & = & b(cc^{-1}) & \text{by \_\_\_.} \\ \text{\_\_\_\_} & = & \text{\_\_\_\_} & \text{by \_\_\_.} \\ a & = & b & \text{by M3.} \end{array}$$

3.1(vi) $ab = 0 \implies a = 0$ or $b = 0$.

Let $ab = 0$ and suppose $a \neq 0$ and $b \neq 0$. Thus, a^{-1} and b^{-1} exist by ____.

$$\begin{array}{rcll} ab(b^{-1}) & = & 0 \cdot b^{-1} & \text{by def. of \_\_\_\_\_\_.} \\ a(bb^{-1}) & = & 0 \cdot b^{-1} & \text{by M1.} \\ a(bb^{-1}) & = & 0 & \text{by \_\_\_.} \\ \text{\_\_\_\_} & = & 0 & \text{by \_\_\_.} \\ a & = & 0 & \text{by \_\_\_.} \end{array}$$

But $a = 0$ contradicts our supposition. Thus, if $ab = 0$ then $a = 0$ or $b = 0$.

3.2(i) $a \leq b \implies -b \leq -a$.

$$\begin{array}{rcll} a & \leq & b & \text{is given.} \\ a + ((-a) + (-b)) & \leq & b + ((-a) + (-b)) & \text{by O4 on both sides.} \\ (a + (-a)) + (-b) & \leq & b + ((-b) + (-a)) & \text{by \_\_\_ \& \_\_\_.} \\ 0 + (-b) & \leq & (b + (-b)) + (-a) & \text{by \_\_\_ \& \_\_\_.} \\ -b & \leq & 0 + (-a) & \text{by \_\_\_ \& \_\_\_.} \\ -b & \leq & -a & \text{by A3.} \end{array}$$

3.2(ii) $a \leq b$ & $c \leq 0 \implies bc \leq ac$.

I'll need to use that $-0 = 0$ so I'll prove that first.

$$\begin{array}{rcl}
(i) & 0 + (-0) = 0 & \text{by A4.} \\
(ii) & 0 = 0 + 0 & \text{by A3.} \\
(i)\&(ii) & \implies 0 + (-0) = 0 + 0. & \\
\text{Thus,} & -0 = 0 & \text{by Thm 3.1___.}
\end{array}$$

Assume $a \leq b$ and $c \leq 0$. Since $c \leq 0$, 3.2(i) implies $-0 \leq -c$. Thus, $0 \leq -c$.

$$\begin{array}{rcll}
a(-c) & \leq & b(-c) & \text{by ___} \\
(-c)a & \leq & (-c)b & \text{by ___.} \\
-(ca) & \leq & -(cb) & \text{by ____.} \\
-(ac) & \leq & -(bc) & \text{by M2} \\
bc & \leq & ac & \text{by ____.}
\end{array}$$

3.2(iii) $0 \leq x$ & $0 \leq y \implies 0 \leq xy$.

Apply O5 with $a = 0$, $b = x$ and $c = y$.

3.2(iv) $0 \leq a^2$.

Either $a \leq 0$ or $0 \leq a$ by O1.

Suppose $0 \leq a$. Then $0 \leq a^2$ by 3.2(iii) with $a = b$.

Suppose $a \leq 0$. By 3.2(i) we have $0 \leq -a$. Thus, $0 \leq (-a)^2$.

Since $(-a)^2 = a^2$ by 3.1(iv) we have $0 \leq a^2$.

More! Fill in the gaps in the proofs of the following addition facts.

Theorem.
(a) $-a = -1 \cdot a$.
(b) see thm 3.1(iv)
(c) $0 < 1$.
(d) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.
(e) If $a < b < 0$, then $b^{-1} < a^{-1} < 0$.

(f) If $0 < a < b$, then $0 < a^2 < b^2$.
(g) If $a < b < 0$, then $0 < b^2 < a^2$.
(h) If $a < b$, then $a^3 < b^3$.

Proof of (a). Hint: We know that of $a + x = a + y$ then $x = y$ by Thm 3.1(i) and A2. We know that $a + (-a) = 0$. If you can show that $a + (-1 \cdot a) = 0$, then you have

$$a + (-a) = 0 = a + (-1 \cdot a).$$

Now use Thm 3.1(i).

Proof of (c). Use Thm 3.2(iv) to get $0 \leq 1^2$. Now suppose $0 = 1$. Let $a \in \mathbb{F}$. Show that $a = 0$. Thus, $\mathbb{F}$ has only one member, which contradicts our defintion of a field.

Proof of (d). Do this one on your own.

Proof of (e). This is similar to (c) but remember multiplying by a negative switches the inequlities.

Proof of (f). Hint: Starting with $0 < a < b$ multiple through by a. Starting with $0 < a < b$ multiple through by b. Compare.

Proof of (g). This is similar to (e) but remember multiplying by a negative switches the inequlities.

Proof of (h). Break it down into cases: $0 < a < b$, $a < b < 0$, $a < 0 < b$, $0 = a < b$, and $a < b = 0$. Are those all the cases?