

## Ordered Fields

**Definition.** Let  $S$  be a set. A **binary operation** is a map from  $S \times S$  to  $S$ .

**Definition.** An **ordered field** is a set  $\mathbb{F}$  with more than one member together with two binary operations, addition  $+$  and multiplication  $\cdot$  and an order relation  $\leq$  satisfying the axioms below for all  $a, b$ , and  $c$  in  $\mathbb{F}$ . (The  $\cdot$  is often left unwritten.)

**A1.**  $a + (b + c) = (a + b) + c$ .

**A2.**  $a + b = b + a$ .

**A3.**  $\exists 0 \in \mathbb{F}$  such that  $a + 0 = a$ .

**A4.**  $\exists -a \in \mathbb{F}$  such that  $a + (-a) = 0$ .

**M1.**  $a(bc) = (ab)c$ .

**M2.**  $ab = ba$ .

**M3.**  $a \cdot 1 = a$ .

**M4.** If  $a \neq 0$ ,  $\exists a^{-1} \in \mathbb{F}$  such that  $a \cdot a^{-1} = 1$ .

**DL.**  $a(b + c) = ab + ac$ .

**O1.** Either  $a \leq b$  or  $b \leq a$  is true.

**O2.** If  $a \leq b$  and  $b \leq a$ , then  $a = b$ .

**O3.** If  $a \leq b$  and  $b \leq c$ , then  $a \leq c$ .

**O4.** If  $a \leq b$ , then  $a + c \leq b + c$ .

**O5.** If  $a \leq b$  and  $0 \leq c$ , then  $ac \leq bc$ .

Note: Given  $\leq$  then  $\geq$ ,  $<$  and  $>$  are defined as usual.

From these beginnings the following can be proven.

**Theorem 3.1** Let  $\mathbb{F}$  be a field. Then  $\forall a, b, c$  in  $\mathbb{F}$  the following hold.

- (i) If  $a + c = b + c$ , then  $a = b$ .
- (ii)  $a \cdot 0 = 0$ .
- (iii)  $(-a)b = -(ab)$ .
- (iv)  $(-a)(-b) = ab$ .
- (v) If  $ac = bc$  and  $c \neq 0$ , then  $a = b$ .
- (vi) If  $ab = 0$  then either  $a = 0$  or  $b = 0$ .

**Theorem 3.2** Let  $\mathbb{F}$  be an ordered field. Then  $\forall a, b, c$  in  $\mathbb{F}$  the following hold.

- (i) If  $a \leq b$ , then  $-b \leq -a$ .
- (ii) If  $a \leq b$  and  $c \leq 0$ , then  $bc \leq ac$ .
- (iii) If  $0 \leq a$  and  $0 \leq b$ , then  $bc \leq ac$ .
- (iv)  $0 \leq a^2$ .
- (v)  $0 < 1$ .
- (vi) If  $0 < a$ , then  $0 < a^{-1}$ .
- (vii) If  $0 < a < b$ , then  $0 < b^{-1} < a^{-1}$ .

**Some Proofs.****3.1(i)** If  $a + c = b + c$ , then  $a = b$ .

$$\begin{array}{ll}
a + c = b + c & \text{is given.} \\
(a + c) + (-c) = (b + c) + (-c) & \text{by def. of binary op.} \\
a + (c + (-c)) = b + (c + (-c)) & \text{by axiom A1 applied to both sides.} \\
a + 0 = b + 0 & \text{by A4.} \\
a = b & \text{by A3.}
\end{array}$$

**3.1(ii)**  $a \cdot 0 = 0$ .

$$\begin{array}{ll}
a \cdot 0 = a(0 + 0) & \text{A3} \\
a \cdot 0 = a \cdot 0 + a \cdot 0 & \text{DL} \\
a \cdot 0 + 0 = a \cdot 0 + a \cdot 0 & \text{A3} \\
0 + a \cdot 0 = a \cdot 0 + a \cdot 0 & \text{A2} \\
0 = a \cdot 0 & \text{3.1(i)}
\end{array}$$

**3.1(iii)**  $(-a)b = -(ab)$ .

$$\begin{array}{ll}
0 \cdot b = 0 & \text{3.1(ii)} \\
(a + (-a))b = 0 & \text{A4} \\
ab + (-a)b = 0 & \text{DL} \\
ab + -(ab) = 0 & \text{A4} \\
ab + -(ab) = ab + (-a)b & \text{since } 0=0 \\
(-a)b = -(ab) & \text{3.1(i)}
\end{array}$$

**3.1(iv)**  $(-a)(-b) = ab$ .

It will be useful to first prove that  $-(-x) = x$ . Since  $(-x) + x = 0$  and  $(-x) + (-(-x)) = 0$ , Theorem 3.1(i) says that  $-(-x) = x$ .

$$\begin{array}{ll}
(-a)(-b) = -(a(-b)) & \text{3.1(iii)} \\
= -((-b)a) & \text{M2} \\
= -(-(-ba)) & \text{3.1(iii)} \\
= ab
\end{array}$$

**3.1(v)**  $ac = bc$  &  $c \neq 0 \implies a = b$ .

First,  $c^{-1}$  exists by M4.

$$\begin{aligned} (ac)c^{-1} &= (bc)c^{-1} \\ a(cc^{-1}) &= b(cc^{-1}) && \text{M1} \\ a \cdot 1 &= b \cdot 1 && \text{M4} \\ a &= b && \text{M3} \end{aligned}$$

**3.1(vi)**  $ab = 0 \implies a = 0$  or  $b = 0$ .

Let  $ab = 0$  and suppose  $a \neq 0$  and  $b \neq 0$ . Thus,  $a^{-1}$  and  $b^{-1}$  exist.

$$\begin{aligned} ab(b^{-1}) &= 0 \cdot b^{-1} \\ a(bb^{-1}) &= 0 && \text{M1 \& 3.1(ii)} \\ a \cdot 1 &= 0 && \text{M4} \\ a &= 0 && \text{M3} \end{aligned}$$

But  $a = 0$  contradicts our supposition. Thus, if  $ab = 0$  then  $a = 0$  or  $b = 0$ .

**3.2(i)**  $a \leq b \implies -b \leq -a$ .

$$\begin{aligned} a &\leq b && \text{given} \\ a + ((-a) + (-b)) &\leq b + ((-a) + (-b)) && \text{O4} \\ (a + (-a)) + (-b) &\leq b + ((-b) + (-a)) && \text{A1 \& A2} \\ 0 + (-b) &\leq (b + (-b)) + (-a) && \text{A4 \& A1} \\ -b &\leq 0 + (-a) && \text{A3 \& A4} \\ -b &\leq -a && \text{A3} \end{aligned}$$

**3.2(ii)**  $a \leq b \ \& \ c \leq 0 \implies bc \leq ac$ .

I'll need to use that  $-0 = 0$  so I'll prove that first.

$$(i) \quad 0 + (-0) = 0 \quad \text{by A4.}$$

$$(ii) \quad 0 = 0 + 0 \quad \text{by A3.}$$

$$(i)\&(ii) \implies 0 + (-0) = 0 + 0.$$

$$\text{Thus,} \quad -0 = 0 \quad \text{by Thm 3.1(i).}$$

Assume  $a \leq b$  and  $c \leq 0$ . Since  $c \leq 0$ , 3.2(i) impels  $-0 \leq -c$ . Thus,  $0 \leq -c$ .

$$\begin{array}{lll} a(-c) & \leq & b(-c) \quad \text{O5} \\ -(ac) & \leq & -(bc) \quad \text{M2,3.1(iii),M2} \\ bc & \leq & ac \quad \text{3.2(i)} \end{array}$$

**3.2(iii)**  $0 \leq x \ \& \ 0 \leq y \implies 0 \leq xy$ .

Apply O5 with  $a = 0$ ,  $b = x$  and  $c = y$ .

**3.2(iv)**  $0 \leq a^2$ .

Either  $a \leq 0$  or  $0 \leq a$  by O1.

Suppose  $0 \leq a$ . Then  $0 \leq a^2$  by 3.2(iii) with  $a = b$ .

Suppose  $a \leq 0$ . By 3.2(i) we have  $0 \leq -a$ . Thus,  $0 \leq (-a)^2$ .

Since  $(-a)^2 = a^2$  by 3.1(iv) we have  $0 \leq a^2$ .