Homework for Math 352
Spring 2020

Hwk Set 3. Due Monday, January 27.

1. Using only the properties of an ordered field prove the following.
   (a) $-a = -1 \cdot a$.
   (b) $(-a)(-b) = ab$.
   (c) $0 < 1$. Hint: Show that if $0 = 1$ then the "field" has only one member.
   (d) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.
   (e) If $a < b < 0$, then $b^{-1} < a^{-1} < 0$.
   (f) If $0 < a < b$, then $0 < a^2 < b^2$.
   (g) If $a < b < 0$, then $0 < b^2 < a^2$.
   (h) If $a < b$, then $a^3 < b^3$.
2. (Bonus!) (a) Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be vectors in $\mathbb{R}^2$. Recall that for vectors the magnitude is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$. Prove that the triangle inequality holds for vectors in $\mathbb{R}^2$, that is
$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|.$$
   (b) Draw a picture illustrating this concept.
3. For each set below find the sup and inf; here $\pm\infty$ are allowed. Regard these as subsets of the real numbers.
   (a) The domain of $\ln x$.
   (b) The range of e^x.
   (c) The range of $e^x \sin x$.
   (d) The range of $\frac{e^x+5}{3e^x+2}$.
   (e) The solution set of $\sin x = 0$.
   (f) The solution set of $\sin\left(\frac{1}{x}\right) = 0$.
   (g) The solution set of $\frac{x}{|x|} \sin^2\left(\frac{1}{x} + x\right) = \frac{1}{2}$.
   (h) Rational numbers r such that $r^2 < 5$.
   (i) Rational numbers r such that $r^3 < 5$.
   (j) $\{2 - \frac{1}{n} \mid n = 1, 2, 3, \ldots\}$.
4. (Bonus!) Let p and q be prime natural numbers. Prove that $\sqrt{p} + \sqrt{q}$ is not rational.
5. (Bonus!) Let $\mathbb{Z}_n$ denote the set $\{0, 1, 2, \ldots, n-1\}$. On $\mathbb{Z}_n$ we use addition and multiplication modulo n.
   (a) Set up addition and multiplication tables for $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_5$.
   (b) Show that $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$ satisfy the field axioms: A1, A2, A3, A4, M1, M2, M3, M4, and DL. (They are not given an order relation.)
   (c) Show that $\mathbb{Z}_4$ satisfies A1, A2, A3, A4, M1, M2, M3, DL, but not M4.
   Note: It can be shown that $\mathbb{Z}_n$ is a field if and only if n is prime. You will see this in MATH 419.