1. Using only the properties of an ordered field prove the following.
   (a) \(-a = -1 \cdot a\).
   (b) \((-a)(-b) = ab\).
   (c) \(0 < 1\). Hint: Show that if \(0 = 1\) then the “field” has only one member.
   (d) If \(0 < a < b\), then \(0 < b^{-1} < a^{-1}\).
   (e) If \(a < b < 0\), then \(b^{-1} < a^{-1} < 0\).
   (f) If \(0 < a < b\), then \(0 < a^2 < b^2\).
   (g) If \(a < b < 0\), then \(0 < b^2 < a^2\).
   (h) If \(a < b\), then \(a^3 < b^3\).

2. (Bonus!) (a) Let \(a = \langle a_1, a_2 \rangle\) and \(b = \langle b_1, b_2 \rangle\) be vectors in \(\mathbb{R}^2\). Recall that for vectors the magnitude is given by 
   \(|a| = \sqrt{a_1^2 + a_2^2}\). Prove that the triangle inequality holds for vectors in \(\mathbb{R}^2\), that is
   \(|a + b| \leq |a| + |b|\).
   (b) Draw a picture illustrating this concept.

3. For each set below find the sup and inf; here \(\pm \infty\) are allowed. Regard these as subsets of the real numbers.
   (a) The domain of \(\ln x\).
   (b) The range of \(e^x\).
   (c) The range of \(e^x \sin x\).
   (d) The range of \(\frac{e^x + 5}{3e^x + 2}\).
   (e) The solution set of \(\sin x = 0\).
   (f) The solution set of \(\sin \left(\frac{1}{x}\right) = 0\).
   (g) The solution set of \(\frac{x}{|x|} \sin^2 \left(\frac{1}{x} + x\right) = \frac{1}{2}\).
   (h) Rational numbers \(r\) such that \(r^2 < 5\).
   (i) Rational numbers \(r\) such that \(r^3 < 5\).
   (j) \(\{2 - \frac{1}{n} \mid n = 1, 2, 3, \ldots\}\).

4. (Bonus!) Let \(p\) and \(q\) be prime natural numbers. Prove that \(\sqrt{p} + \sqrt{q}\) is not rational.

5. (Bonus!) Let \(Z_n\) denote the set \(\{0, 1, 2, \ldots, n - 1\}\). On \(Z_n\) we use addition and multiplication modulo \(n\).
   (a) Set up addition and multiplication tables for \(Z_2, Z_3, Z_4\) and \(Z_5\).
   (b) Show that \(Z_2, Z_3\) and \(Z_5\) satisfy the field axioms: A1, A2, A3, A4, M1, M2, M3, M4, and DL. (They are not given an order relation.)
   (c) Show that \(Z_4\) satisfies A1, A2, A3, A4, M1, M2, M3, DL, but not M4.
   Note: It can be shown that \(Z_n\) is a field if and only if \(n\) is prime. You will see this in MATH 419.