Sequences with Infinite Limits

Definition. Let (s_n) be an infinite sequence of real numbers.

- $\lim_{n\to\infty} s_n = \infty$ if for every positive real number B there is a natural number N such that $n \geq N$ implies $s_n > B$.
- $\lim_{n\to\infty} s_n = -\infty$ if for every negative real number B there is a natural number N such that $n \geq N$ implies $s_n < B$.

Example. Prove that $\lim_{n\to\infty} n^2 = \infty$.

Proof. Let $B > 0$ be given. There exists a natural number N larger than B. Let $n \geq N$. Then $n^2 \geq N^2 > B^2 > B$. Thus, $\lim_{n\to\infty} n^2 = \infty$. □

Example. Prove that $\lim_{n\to\infty} \dfrac{n^2}{n+1} = \infty$.

Proof. Let $B > 0$ be given. Notice that $\frac{n^2}{n+1} = n - 1 + \frac{1}{n+1}$. Let N be a natural number greater than $B + 1$. Then for $n \geq N$ we have

$$n - 1 + \frac{1}{n+1} > n - 1 \geq N - 1 > B.$$

Thus, $\lim_{n\to\infty} \dfrac{n^2}{n+1} = \infty$. □

There are analogs of some of the theorems for finite limits to infinite limits. The statements are a bit more complex. I'll just list a few of them here.

- If $s_n \to \infty$ and $t_n \to \infty$, then $s_n + t_n \to \infty$.
- If $s_n \to \infty$ and $t_n \to L \in \mathbb{R}$, then $s_n + t_n \to \infty$.
- If $s_n \to \infty$ and (t_n) is bounded, then $s_n + t_n \to \infty$.

- If $s_n \to \infty$ and $t_n \to \infty$, then $s_n t_n \to \infty$.
- If $s_n \to \infty$ and $t_n \to L > 0$, then $s_n t_n \to \infty$.
- If s_n is never zero and $s_n \to \infty$, then $1/s_n \to 0$.
- If $s_n \to \infty$ and $t_n > s_n$, then $t_n \to \infty$.

Similar statements hold for limits to $-\infty$. However, if $s_n \to \infty$ and $t_n \to -\infty$, no general conclusion can be drawn about the limit of $(s_n + t_n)$. If $s_n \to \infty$ and $t_n \to 0$, no general conclusion can be drawn about the limit of $(s_n t_n)$.

Here is an application.

Theorem. Let $p(x) = a_m x^m + \cdots + a_1 x + a_0$ and $q(x) = b_k x^k + \cdots + b_1 x + b_0$ be polynomials with real coefficients; assume $a_m \neq 0$ and $b_k \neq 0$.

- If $m > k$, then $\lim_{n\to\infty} \dfrac{p(n)}{q(n)} = \pm\infty$, where the sign is the same as the sign of a_m.
- If $m < k$, then $\lim_{n\to\infty} \dfrac{p(n)}{q(n)} = 0$.
- if $m = k$, then $\lim_{n\to\infty} \dfrac{p(n)}{q(n)} = \dfrac{a_m}{b_k}$.