

Summary of the properties of limits of functions of a real variable

Finite Limit Laws. Let c and a be real numbers (constants). Assume that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist and are finite. Then the following hold.

- (1) $\lim_{x \rightarrow a} c = c$.
- (2) $\lim_{x \rightarrow a} x = a$.
- (3) $\lim_{x \rightarrow a} c f(x) = c \lim_{x \rightarrow a} f(x)$.
- (4) $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$.
- (5) $\lim_{x \rightarrow a} f(x)g(x) = \left(\lim_{x \rightarrow a} f(x)\right) \left(\lim_{x \rightarrow a} g(x)\right)$.
- (6) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided $\lim_{x \rightarrow a} g(x) \neq 0$.
- (7) $\lim_{x \rightarrow a} (f(x))^c = \left(\lim_{x \rightarrow a} f(x)\right)^c$, unless $\lim_{x \rightarrow a} f(x) = 0$ and $c < 0$.

Analogous statements are true if we replace $x \rightarrow a$ with $x \rightarrow a^+$, $x \rightarrow a^-$ or $x \rightarrow \pm\infty$.

Infinite Limit Laws. Let a , $L \neq 0$ and $c \neq 0$ be real constants. Let p, q, n, z , and k functions such that $\lim_{x \rightarrow a} p(x) = \infty$, $\lim_{x \rightarrow a} q(x) = \infty$, $\lim_{x \rightarrow a} z(x) = 0$ and $\lim_{x \rightarrow a} k(x) = L$. Then the following hold.

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| (1) $\lim_{x \rightarrow a} p(x) + q(x) = \infty$ | (2) $\lim_{x \rightarrow a} p(x) \pm k(x) = \infty$ |
| (3) $\lim_{x \rightarrow a} -p(x) \pm k(x) = -\infty$ | (4) $\lim_{x \rightarrow a} p(x)q(x) = \infty$ |
| (5) $\lim_{x \rightarrow a} -p(x)q(x) = -\infty$ | (6) $\lim_{x \rightarrow a} cp(x) = \text{sign}(c) \infty$ |
| (7) $\lim_{x \rightarrow a} k(x)p(x) = \text{sign}(L) \infty$ | (8) $\lim_{x \rightarrow a} \frac{1}{p(x)} = 0$ |
| (9) No conclusion can be drawn for $\lim_{x \rightarrow a} p(x)z(x)$ or $\lim_{x \rightarrow a} p(x) - q(x)$. | |

Analogous statements are true if we replace $x \rightarrow a$ with $x \rightarrow a^+$, $x \rightarrow a^-$ or $x \rightarrow \pm\infty$.

The infinite limit laws may be abbreviated as follows.

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| (1) $\infty + \infty = \infty$ | (2) $\infty \pm L = \infty$ | |
| (3) $-\infty \pm L = -\infty$ | (4) $\infty \cdot \infty = \infty$ | |
| (5) $-\infty \cdot \infty = -\infty$ | (6&7) $c \infty = \text{sign}(c) \infty$ | (8) $\frac{1}{\infty} = 0$ |

The Removable Singularity Rule. Suppose $g(x)$ is continuous on (a, c) and that $f(x) = g(x)$ on an $(a, b) \cup (b, c)$. Then $\lim_{x \rightarrow b} f(x) = g(b)$.

The Composition Theorem. If $\lim_{x \rightarrow a} g(x) = P$ and $\lim_{y \rightarrow P} f(y) = L$ then $\lim_{x \rightarrow a} f(g(x)) = L$. This holds true when any of a , P or L are infinities. If f is continuous at P then this can be written as

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(L).$$

The Squeeze Theorem. Suppose $f(x) \leq g(x) \leq h(x)$ on a suitable domain. Then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \leq \lim_{x \rightarrow a} h(x)$, provided the limits exist. If $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$ then $\lim_{x \rightarrow a} g(x) = L$. This holds when $L = \pm\infty$ and for limits as $x \rightarrow \pm\infty$ or one sided limits. (The reader should be able to determine what is meant by a *suitable domain*.)