

# Integration by Parts <sup>1</sup>

## The Product Rule in Reverse

Unfortunately there is no systematic procedure for integrating the product of two functions with known anti-derivatives. However, the product rule for derivatives does yield a useful trick for doing certain integrals. The method is called **Integration by Parts**. Let  $u$  and  $v$  be functions of  $x$ ; but we will also think of them as variables in their own rights. The product rule states,

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.$$

Integrate both sides.

$$\int \frac{d(uv)}{dx} dx = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx.$$

By the Substitution Theorem ,

$$\int duv = \int u dv + \int v du,$$

or,

$$uv + C = \int u dv + \int v du.$$

This is normally rewritten as

$$\boxed{\int u dv = uv - \int v du.}$$

(We drop the  $C$  since both sides have indefinite integrals.)

What is this good for? We do some examples.

**Example 1.** Find  $\int x \cos x dx$ .

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*Solution.* Let  $u = x$  and  $dv = \cos x \, dx$ . Then  $du = dx$  and  $v = \sin x$ . Now

$$\begin{aligned}\int x \cos x \, dx &= \int u \, dv \\ &= uv - \int v \, du \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C\end{aligned}$$

□

We did not need to set  $v = \sin x + C$  since all that we require is the  $v$  be some anti-derivative of  $\cos x$ . The key is making the right choices for  $u$  and  $v$ . We used  $u = x$  since the derivative of  $x$  is simpler than  $x$  and the integral of  $\cos x$  does not get any “worse”. Try redoing Example ?? with  $u = \cos x$  and  $dv = dx$  and see what happens. However, there is not a hard and fast rule for how to chose  $u$  and  $dv$ .

**Example 2.** Find  $\int \ln x \, dx$ .

*Solution.* Let  $u = \ln x$  and  $dv = dx$ . Then  $du = \frac{1}{x} dx$  and  $v = x$ . Thus,

$$\begin{aligned}\int \ln x \, dx &= \int u \, dv \\ &= uv - \int v \, du \\ &= x \ln x - \int x \frac{1}{x} \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C\end{aligned}$$

□

**Example 3.** Find  $\int e^x \sin x \, dx$ .

*Solution.* This will be a two step process. Let  $u = e^x$  and  $dv = \sin x dx$ . Then  $du = e^x dx$  and  $v = -\cos x$ . Thus,

$$\begin{aligned}\int e^x \sin x dx &= -\int u dv \\ &= -uv + \int v du \\ &= -e^x \cos x + \int e^x \cos x dx\end{aligned}\tag{1}$$

But now what to do? We apply integration by parts to the last integral. This, as you will check, gives

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

By substituting into equation (1) we get,

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx.$$

(Now for a trick!) This last equation implies,

$$2 \int e^x \sin x dx = -e^x \cos x + e^x \sin x + C,$$

or

$$\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2} + C.$$

You may need to reread this example several times, working out the steps on your own.  $\square$

**Example 4.** Find  $I = \int \sec^3 x dx$ .

*Solution.* It is not obvious how to start. Since the integral  $\sec^2 x$  is easy we will try  $u = \sec x$  and  $dv = \sec^2 x dx$ . Then  $du = \sec x \tan x dx$  and  $v = \tan x$ . Thus,

$$I = \int \sec x \sec^2 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec^3 x - \sec x dx.$$

Thus,

$$I = \sec x \tan x - I + \ln |\sec x + \tan x| + C.$$

And finally,

$$I = \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C.$$

□

**Example 5.** Find  $\int x^2 \sin x \, dx$ .

*Solution.* Taking a derivative of  $x^2$  makes it simpler and integrating the sine function won't make matters worse. So, we let  $u = x^2$  and  $dv = \sin x \, dx$ . Then  $du = 2x \, dx$  and  $v = -\cos x$ . Thus,

$$\int x^2 \sin x \, dx = \int u \, dv = uv - \int v \, du = -x^2 \cos x + 2 \int x \cos x \, dx.$$

This is a step in the right direction. Now let  $u = x$  and  $dv = \cos x \, dx$ . Then  $du = dx$  and  $v = \sin x$ . Thus,

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Combining these results we get

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

You can check the result by taking the derivative. □

**Example 6.** Find  $\int x^3 e^x \, dx$ .

*Solution.* Let  $u = x^3$  and  $dv = e^x \, dx$ . Then  $du = 3x^2 \, dx$  and  $v = e^x$ . Thus,

$$\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx.$$

We repeat. Let  $u = x^2$  and  $dv = e^x \, dx$ . Then  $du = 2x \, dx$  and  $v = e^x$ . Thus,

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

Once more. Let  $u = x$  and  $dv = e^x \, dx$ . Then  $du = dx$  and  $v = e^x$ . Thus,

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Therefore,

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C.$$

□

This leads to the idea of a **reduction formula**.

**Example 7.** Show that  $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$ . This is an example of a reduction formula.

*Solution.* Let  $u = x^n$  and  $dv = e^x dx$ . Then  $du = nx^{n-1}$  and  $v = e^x$ . Therefore,

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

□

**Example 8.** Find  $\int x^6 e^x dx$ .

*Solution.*

$$\begin{aligned} \int x^6 e^x dx &= x^6 e^x - 6 \left( \int x^5 e^x dx \right) \\ &= x^6 e^x - 6x^5 e^x + 6 \cdot 5 \left( \int x^4 e^x dx \right) \\ &= x^6 e^x - 6x^5 e^x + 6 \cdot 5x^4 e^x - 6 \cdot 5 \cdot 4 \left( \int x^3 e^x dx \right) \\ &= x^6 e^x - 6x^5 e^x + 6 \cdot 5x^4 e^x - 6 \cdot 5 \cdot 4x^3 e^x + 6 \cdot 5 \cdot 4 \cdot 3 \left( \int x^2 e^x dx \right) \\ &= x^6 e^x - 6x^5 e^x + 6 \cdot 5x^4 e^x - 6 \cdot 5 \cdot 4x^3 e^x + 6 \cdot 5 \cdot 4 \cdot 3x^2 e^x - 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \left( \int x e^x dx \right) \\ &= x^6 e^x - 6x^5 e^x + 6 \cdot 5x^4 e^x - 6 \cdot 5 \cdot 4x^3 e^x + 6 \cdot 5 \cdot 4 \cdot 3x^2 e^x - 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2x e^x + 6! e^x + \end{aligned}$$

□

Note.  $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$ .

We can now conclude that

$$\int x^n e^x dx = (-1)^n e^x \sum_{k=1}^n (-1)^k \frac{n! x^k}{k!} + C.$$

#### PROBLEMS.

$2 \int x \sin 3x dx$   $\int x^2 e^x dx$   $\int x \ln x dx$   $\int x^2 \ln x dx$   $\int e^x \cos x dx$   $\int e^{2x} \cos 3x dx$   
 $\int e^{ax} \sin bx dx$   $\int \arctan x dx$   $\int \sqrt{x} \ln x dx$   $\int x \csc^2 x dx$  Find reduction  
 formulas for  $\int x^n \sin x dx$  and  $\int x^n \cos x dx$ . Find  $\int x^{11} \sin x dx$ .