

## Power Series

Let  $f_n : D \rightarrow \mathbb{R}$  for  $n = 0, 1, 2, 3, \dots$  be functions. We study infinite sequences and series of functions.

For now we just do **power series** which are series of functions of the form

$$\sum_{n=p}^{\infty} a_n x^n$$

where usually  $p = 0$  or  $1$ .

(If all the  $a_n$  were the same value we would have a geometric series.)

You may recall from Calculus that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x,$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sin x.$$

**Theorem.** Consider  $\sum_{n=p}^{\infty} a_n x^n$ . Let

$$R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}}.$$

Then  $R$  is called the **radius of convergence** and

- (i) the series converges if  $|x| < R$ ,
- (ii) the series diverges if  $|x| > R$ .

No conclusion can be drawn when  $|x| = R$ ; one has to *check the end points*. Note that if  $R = \infty$  the series converges for all  $x$ . The series always converges for  $x = 0$ .

The proof uses the Root Test. The version of the Root Test we did in class used only the limit of  $\sqrt[n]{|a_n|}$  rather than the limit superior done in the textbook. The proof both cases is the same. The reason we need the stronger version we be clear when we do examples.

*Proof.* Case 1. Suppose  $0 < R < \infty$ . Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n x^n|} = \frac{|x|}{R}.$$

For  $|x| < R$  the series converges by the Root Test and for  $|x| > R$  the series diverges by the Root Test.

Case 2. Suppose  $R = 0$ . Then for  $x \neq 0$

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n x^n|} = \infty.$$

Hence Root Test implies the series diverges.

Case 3. Suppose  $R = \infty$ . Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n x^n|} = 0 < 1$$

and the Root Test implies the series converges.  $\square$

In problems where the limit of the ratio  $a_{n+1}/a_n$  exists it is equal to  $\sqrt[n]{|a_n x^n|}$  by Corollary 12.3, so you can just use the Ratio Test. Example 6 in the textbook shows what can happen when this fails - we will discuss it in class.

**Example.** Find the radius of convergence of  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ .

*Proof.* Every odd term has  $a_n = 0$ . Thus,  $\liminf \sqrt[n]{|a_n x^n|} = 0$ . The lim sup is the limit of the even terms.

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n x^n|} = \lim_{n \rightarrow \infty} \sqrt[2n]{|a_{2n} x^{2n}|} = |x| \lim_{n \rightarrow \infty} \frac{1}{\sqrt[2n]{(2n)!}}.$$

Now this is a subsequence of  $\left(\frac{1}{\sqrt[n]{(n)!}}\right)$ . This limit was Exercise 12.14a. It uses Corollary 12.3. Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{(n)!}} = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0.$$

Thus,  $R = \infty$ . □