Series

Definition. The infinite sum is defined to
\[ \sum_{k=p}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=p}^{n} a_k \]
when the limit exists. Usually the sum will start with \( k = 1 \) or 0.

The n-th partial sum of \( \sum_{k=p}^{\infty} a_k \) is \( s_n = \sum_{k=p}^{p+n-1} a_k \).

Now we can write \( \sum_{k=p}^{\infty} a_k = \lim_{n \to \infty} s_n \) when the limit exists.

Note. If we leave off the first few terms of a sequence the limit is unaffected. This is a special case of the subsequence theorem. If we leave off a finite number of terms of an infinite sum, then whether it converges or not is unaffected, but if it converges the value will be changed.

A series converges if and only if the sequence of partial sums is Cauchy. Notice that for \( n > m \)
\[ s_n - s_m = \sum_{k=m+1}^{n} a_n. \]

We say a series satisfies the Cauchy criteria if \( \forall \epsilon > 0, \exists N \in \mathbb{N} \) s.t.
\[ n \geq m > N \implies \left| \sum_{k=m}^{n} a_k \right| < \epsilon. \]

Thus, a series converges if and only if it satisfies the Cauchy criteria. (This is Theorem 14.4 in the textbook.)
Corollary (14.5). If \( \sum a_k \) converges then \( a_k \to 0 \).

Proof. Let \( \epsilon > 0 \). \( \exists N \in \mathbb{N} \) s.t.

\[
    n \geq m > N \implies \left| \sum_{k=m}^{n} a_k \right| < \epsilon.
\]

Thus, for \( k > N \) we have \( |a_k| < \epsilon \) by using \( n = m = k \). Thus, \( a_k \to 0 \).

Example. (This is Exercise 4.14 in your textbook.)

\[
    \sum_{n=1}^{\infty} \frac{1}{n} = \infty.
\]

Proof. Let \( s_m = \sum_{n=1}^{m} \frac{1}{n} \). We will show \( s_m \to \infty \). Consider the subsequence \( (s_{m_k})_{k=1}^{\infty} \) where \( m_k = 2^k \).

We will use induction to prove that \( s_{m_k} \geq 1 + \frac{k}{2} \) for all \( k \in \mathbb{N} \). First notice that \( s_{m_1} = s_2 = 1 + \frac{1}{2} \). Suppose, \( s_{m_{k-1}} \geq 1 + \frac{k-1}{2} \) for some \( k > 1 \). Now,

\[
    s_{m_k} = \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}} \right) + \left( \frac{1}{2^k-1+1} + \cdots + \frac{1}{2^k} \right).
\]

Each pair of parentheses has \( 2^{k-1} \) terms. In fact the sum in the first pair is \( s_{m_{k-1}} \). Each term in the second pair before the last term is greater than the last term. Thus,

\[
    s_{m_k} \geq s_{m_{k-1}} + 2^{k-1} \cdot \left( \frac{1}{2^k} \right) = s_{m_{k-1}} + \frac{1}{2}.
\]

By the induction hypothesis \( s_{m_{k-1}} \geq 1 + \frac{k-1}{2} \). Thus,

\[
    s_{m_k} \geq 1 + \frac{k-1}{2} + \frac{1}{2} = 1 + \frac{k}{2}.
\]

Since \( 1 + k/2 \to \infty \) we have that \( s_{m_k} \to \infty \). Thus \( (s_m) \) cannot converge. To show it diverges to infinity we use the fact that \( (s_m) \) is increasing since all the \( \frac{1}{n} \) are positive. Let
$B > 0$. $\exists K \in \mathbb{N}$ s.t.

$$k \geq K \implies s_{mk} > B.$$ 

Thus,

$$m \geq m_k \implies s_m \geq s_{mk} > B.$$ 

Thus, $s_m \to \infty$. 

**Definitions.** A series $\sum a_k$ is called **alternating** if $a_{k+1}$ always has the opposite sign of $a_k$. They come up in many applications. Consider a series $\sum a_n$. If $\sum |a_n|$ converges we say the series is **absolutely convergent**. We will see below that if $\sum |a_n|$ converges then so does $\sum a_n$. If $\sum a_n$ converges but $\sum |a_n|$ does not we say the series is **conditionally convergent**.

There is a slew of tests for convergence. We will run through these and prove a few of them.

**The Direct Comparison Test.** (14.6) Assume $a_n \geq 0$ for all $n \in \mathbb{N}$.

(a) If $\sum a_n$ converges (to a finite real number) and $|b_n| \leq a_n \forall n \in \mathbb{N}$, then $\sum b_n$ converges (to a finite real number $\leq \sum a_n$).

(b) If $\sum a_n = \infty$ and $b_n \geq a_n \forall n \in \mathbb{N}$, then $\sum b_n = \infty$.

(c) Both (a) and (b) hold true if “$\forall n \in \mathbb{N}$” is replaced by “$\forall n > K$ for some $K \in \mathbb{N}$”.

**Proof of (a).** Let $\epsilon > 0$. Let $N$ be s.t. $n > m > N$ implies $\sum_{k=m}^{n} a_k < \epsilon$. Then

$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k < \epsilon.$$ 

You do (b) and (c).
Note. A special case of (a) is that if $\sum |a_n|$ converges then so does $\sum a_n$.

**Theorem.** Assume $\sum a_n$ and $\sum b_n$ converge and let $c \in \mathbb{R}$. Then

(i) $\sum ca_n = c \sum a_n$, and
(ii) $\sum a_n + b_n = \sum a_n + \sum b_n$.

Proofs are easy and left to you. Note that if $\sum a_n$ diverges and $c \neq 0$, then $\sum ca_n$ diverges too.

**The Limit Comparison Test.** (Not in your textbook.) Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \in (0, \infty),$$

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

**Proof.** Let $\epsilon = L/2$. Then $\exists N$ s.t.

$$n > N \implies \frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}.$$

Thus,

$$\frac{Lb_n}{2} < a_n < \frac{3Lb_n}{2}.$$

If $\sum b_n$ converges, then so do $\sum Lb_n/2$ and $\sum 3Lb_n/2$. Thus, $\sum a_n$ converges by the Direct Comparison Test (c).

If a positive term series diverges, it diverges to infinity. If $\sum b_n$ diverges to $\infty$, then so does $\sum Lb_n/2$. By the Direct Comparison Test $\sum a_n$ diverges too.

The other two implications follow logically from these or note that $\lim_{n \to \infty} b_n/a_n = 1/L \in (0, \infty)$. \qed
**Alternating Series Test.** Assume $a_n$ is nonnegative and nonincreasing for all $n$; that is

$$0 \leq a_{n+1} \leq a_n, \quad \forall n \in \mathbb{N}.$$ 

If $a_n \to 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

*Idea of the proof.* See the figure below. It is a bar graph of the partial sums, $s_m = \sum_{n=1}^{m}(-1)^{n+1}a_n$. Notice that the subsequence of odd terms is decreasing and bounded below while the even terms are increasing and bounded above. I’ll discuss it in class.

![Bar graph of partial sums](image)

**Proof.** Let $s_m = \sum_{n=1}^{m}(-1)^{n+1}a_n$. Consider the subsequences $(s_{2k})$ and $(s_{2k-1})$. We will show that following.

(*) The subsequence $(s_{2k})$ is nondecreasing and bounded below. Hence it has a limit $L_e$. 


(**) The subsequence \((s_{2k-1})\) is nonincreasing and bounded above. Hence it has a limit \(L_o\).

(***) Finally, we show \(L_e = L_o\) and that this is the limit of \((s_m)\).

(*) \(s_{2(k+1)} - s_{2k} = -a_{2k+2} + a_{2k+1} \geq 0\) implies \(s_{2(k+1)} \geq s_{2k}\). Thus, \((s_{2k})\) is nondecreasing. We claim \(s_{2k} \leq a_1\) for all \(k\). We write

\[
s_{2k} = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots - a_{2k-2} + a_{2k-1} - a_{2k}
\]

All the terms in parentheses are positive or zero as is the last term. Hence, \(s_{2k} \leq a_1\).

(**) \(s_{2(k+1)-1} - s_{2k-1} = a_{2k+1} - a_{2k} \leq 0\) implies \(s_{2(k+1)-1} \leq s_{2k-1}\). Thus, \((s_{2k})\) is nonincreasing. We claim \(s_{2k-1} \geq a_1 - a_2\) for all \(k\). We write

\[
s_{2k-1} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \cdots a_{2k-3} - a_{2k-2} + a_{2k-1}
\]

All the terms in parentheses are positive or zero as is the last term. Hence, \(s_{2k-1} \geq a_1 - a_2\).

(***) We compute \(L_e - L_o = \)

\[
\lim_{k \to \infty} s_{2k} - \lim_{k \to \infty} s_{2k-1} = \lim_{k \to \infty} s_{2k} - s_{2k-1} = \lim_{k \to \infty} -a_{2k} = 0.
\]

Hence \(L_e = L_o\). Let \(L = L_e\).

Let \(\epsilon > 0\). \(\exists N\) s.t. for \(k > N\) we have \(|s_{2k} - L| < \epsilon\) and \(|s_{2k-1} - L| < \epsilon\). Hence for and \(n > 2N\) we have \(|s_n - L| < \epsilon\). Thus \(s_n \to L\) which is to say

\[
\sum_{n=1}^{\infty} (-1)^{n+1}a_n = L.
\]

\(\square\)
Examples.
The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges. It can be shown that the limit is $\ln 2$.
The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges. It can be shown that the limit is $\pi^2/12$.

Geometric Series Test. A series of the form $\sum ar^k$ is called a geometric series.
(i) If $r \neq 1$, then
\[
\sum_{k=0}^{n} ar^k = a \frac{1 - r^{n+1}}{1 - r}.
\]
(ii) If $|r| < 1$, then
\[
\sum_{k=0}^{n} ar^k = \frac{a}{1 - r}.
\]

Proof. You have done this.

We did some examples in class.

Ratio Test. [This version is the one given in most calculus textbooks. Your textbook gives a jazzed up version that is covered in 452.] Let $\sum a_n$ be an infinite series of nonzero terms.

(i) If $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, then $\sum a_n$ diverges.
Proof. The idea is to use apply the Direct Comparison Theorem using a geometric series.

(i) \( \exists r \in \mathbb{R} \text{ s.t. } L < r < 1. \) Let \( \epsilon = r - L. \) Now, \( \exists N \in \mathbb{N} \) s.t. \( n \geq N \) implies

\[
\left| \frac{a_{n+1}}{a_n} - L \right| < \epsilon.
\]

Thus,

\[-(r - L) < \left| \frac{a_{n+1}}{a_n} \right| - L < r - L.
\]

Thus,

\[
\left| \frac{a_{n+1}}{a_n} \right| < r.
\]

Thus,

\[
|a_{n+1}| < |a_n|r, \ \forall n \geq N.
\]

By induction,

\[
|a_{N+k}| < |a_N|r^k, \ \forall k \in \mathbb{N}.
\]

By the Geometric Series Test

\[
\sum |a_N|r^k
\]
converges since \( |r| < 1. \) By the Direct Comparison Test

\[
\sum_{k=1}^\infty |a_{N+k}|
\]
converges. Therefore,

\[
\sum_{n=1}^\infty |a_n|
\]
converges.

(ii) Similar. \( \square \)
Example. The series \( \sum_{n=0}^{\infty} \frac{1}{n!} \) converges.

Proof. We use the Ratio Test.

\[
\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \to 0 < 1.
\]

\( \Box \)

Later we will show that \( \sum_{n=0}^{\infty} \frac{1}{n!} = e \).

Root Test. [This version is the one given in most calculus textbooks. Your textbook gives a jazzed up version that is covered in 452.] Let \( \sum a_n \) be an infinite series.

(i) If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1 \), then \( \sum a_n \) converges absolutely.

(ii) If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1 \), then \( \sum a_n \) diverges.

Proof. (i) \( \exists r \in \mathbb{R} \) s.t. \( L < r < 1 \). \( \exists N \in \mathbb{N} \) s.t. \( n \geq N \) implies

\[
\sqrt[n]{|a_n|} - L < r - L.
\]

Thus,

\[
\sqrt[n]{|a_n|} < r.
\]

Thus,

\[
|a_n| < r^n.
\]

Thus,

\[
\sum_{n=N}^{\infty} |a_n|
\]
converges by the Direct Comparison Test since $|r| < 1$. It follows that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

You can prove (ii). □

**Integral Test.** See textbook.

*Proof.* See textbook. □

**p-Series Test.** The infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges to $\infty$ for $p \leq 1$.

*Proof.* Use the Integral Test. See textbook. □

**Examples.**

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] converges. In fact it equals $\frac{\pi^2}{6}$.

\[ \sum_{n=1}^{\infty} \frac{1}{n^4} \] converges. In fact it equals $\frac{\pi^4}{90}$.

\[ \sum_{n=1}^{\infty} \frac{1}{n^6} \] converges. In fact it equals $\frac{\pi^6}{945}$.

\[ \sum_{n=1}^{\infty} \frac{1}{n^8} \] converges. In fact it equals $\frac{\pi^4}{9450}$. 