Subsequences

Section 11 deals with subsequences. We will only touch on this section lightly. The main facts you need to know are given below. We will skip Sections 12 and 13. That material is covered in 452.

[Theorem 11.3.] A subsequence of a convergent sequence converges to the same limit. This is also true if the parent sequence diverges to ∞ or $-\infty$.

[Theorem 11.2.] If a sequence has a cluster point, then there is a subsequence that converges to it.

[Theorem 11.5.] This is the famed Bolzano-Weierstrass Theorem. Every bounded sequence has a convergent subsequence.

Other material in Section 11 is optional reading.

Example. Let $(a_n) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots)$. Then $(a_{2n}) = (2, 4, 6, 8, 10, 12, 14, \ldots)$ is a subsequence.

Definition. Let $a : \mathbb{N} \to \mathbb{R}$ be given. Then as usual we let $a_n = a(n)$ and write $(a_n)_{n=1}^{\infty}$ for the sequence. Now let $\sigma : \mathbb{N} \to \mathbb{N}$ be an increasing function. Then $a(\sigma(k))$ maps $\mathbb{N}$ into $\mathbb{R}$ determines a **subsequence**. It is commonly written as $(a_{n_k})_{k=1}^{\infty}$.

Examples. Let (a_n) be a sequence.
Let $n_k = 2k$. This give the subsequence $(a_2, a_4, a_6, \ldots)$.
Let $n_k = 2k-1$. This gives the subsequence $(a_1, a_3, a_5, \ldots)$.
Let $n_k = k^2+13$. This gives the subsequence $(a_{14}, a_{17}, a_{21}, a_{29}, \ldots)$.
Let $n_k = k!$. This gives the subsequence $(a_1, a_2, a_6, a_{24}, a_{120}, \ldots)$.

Theorem. (11.3) If $\lim_{n\to\infty} a_n = L$ then $\lim_{k\to\infty} a_{n_k} = L$ for any subsequence.

Proof. We will only prove this for $L \in \mathbb{R}$ although the result holds if $L = \pm\infty$. Let $\epsilon > 0$.

$\exists N$ s.t. $n > N$ implies $|a_n - L| < \epsilon$.

$\exists K$ s.t. $k > K$ implies $n_k > N$.

Thus, for $k > K$ we have $|a_{n_k} - L| < \epsilon$.

Hence $\lim_{k\to\infty} a_{n_k} = L$. □

Theorem. (Similar to 11.2) Let (a_n) be a sequence and let $c \in \mathbb{R}$ be a cluster point. Then there is a subsequence (a_{n_k}) that converges to c.

Proof. Recall the definition of a cluster point: A number $c \in \mathbb{R}$ is a **cluster point** of (a_n) if $\forall\, \epsilon > 0$ and $\forall N \in \mathbb{N}$ $\exists n > N$ such that

$$|a_n - c| < \epsilon.$$

We shall construct a subsequence as follows.

Let $\epsilon = 1$. Then $\exists n_1 \in \mathbb{N}$ s.t. $|a_{n_1} - c| < 1$.

Let $\epsilon = \frac{1}{2}$. Then $\exists n_2 > n_1$ in $\mathbb{N}$ s.t. $|a_{n_2} - c| < \frac{1}{2}$.

Let $\epsilon = \frac{1}{3}$. Then $\exists n_3 > n_2$ in $\mathbb{N}$ s.t. $|a_{n_3} - c| < \frac{1}{3}$.

Let $\epsilon = \frac{1}{4}$. Then $\exists n_4 > n_3$ in $\mathbb{N}$ s.t. $|a_{n_4} - c| < \frac{1}{4}$.

Continue is this way. Once $n_k > n_{k-1}$ has been chosen such that $|a_{n_k} - c| < \frac{1}{k}$ and we can choose $n_{k+1} > n_k$ such that $|a_{n_{k+1}} - c| < \frac{1}{k+1}$.

Thus, we have a subsequence $(a_{n_k})_{k=1}^{\infty}$. Now we show $a_{n_k} \to c$. Let $\epsilon > 0$ be arbitrarily chosen. Let $K \in \mathbb{N}$ be s.t. $K > 1/\epsilon$. Then for all $k > K$ we have

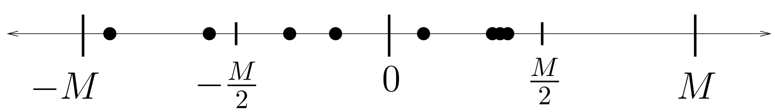
$$|a_{n_k} - c| < 1/k < 1/K < \epsilon.$$

Thus, $a_{n_k} \to c$. $\square$

The Bolzano-Weiestrass Theorem (11.5). Every bounded infinite sequence of real numbers has a convergent subsequence.

Proof. Let (a_n) be an infinite sequence of real numbers bounded by $M > 0$, that is $|a_n| < M$ for all $n \in \mathbb{N}$.

Let $I_1 = [-M, 0]$ and $I_2 = [0, M]$.
Let $I_{11} = [-M, -M/2]$, $I_{12} = [-M/2, 0]$, $I_{21} = [0, M/2]$ and $I_{22} = [M/2, M]$.
Continuing let $I_{111} = [-M, -3M/4]$, $I_{112} = [-3M/4, -M/2]$, etc.
Continue this process indefinitely.



At least one of I_1 and I_2 contains a_n for infinitely many n. Say is it I_i. Let n_1 be the smallest natural number s.t. $a_{n_1} \in I_i$.

At least one of I_{i1} and I_{i2} contains a_n for infinitely many n. Say is it I_{ij}. Let n_2 be the smallest natural number greater than n_1 s.t. $a_{n_2} \in I_{ij}$.

At least one of I_{ij1} and I_{ij2} contains a_n for infinitely many n. Say is it I_{ijk}. Let n_3 be the smallest natural number greater than n_2 s.t. $a_{n_3} \in I_{ijk}$.

This process can be continued indefinitely to generate a subsequence (a_{n_k}). We claim it is Cauchy. Let $\epsilon > 0$.

$\exists\ p$ s.t. $2M \cdot 2^{-p} < \epsilon$. Let $I_{i_1 i_2 i_3 \cdots i_p}$ be the interval that contains a_{n_k} for all $k \geq p$. Then the distance between any two of these is less than $2M \cdot 2^{-p}$ which is less that ϵ. Thus, we have found a subsequence that is Cauchy.

Thus a_{n_k} convergences. Note that its limit will be a cluster point of (a_n). □

Note. The limit of (x_{n_k}) is in $[-M, M]$. *Proof.* Suppose $x_{n_k} \to c > M$. Let $\epsilon = (M - c)/2$. Then x_{n_k} is never in $(c - \epsilon, c + \epsilon)$. A similar contradiction arises if $c < -M$.