

9.3 Theorem Suppose $s_n \rightarrow s$ and $t_n \rightarrow t$ with $s, t \in \mathbb{R}$.
Then $s_n + t_n \rightarrow s + t$.

pf Let $\varepsilon > 0$.

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}.$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}.$$

Let $N = \max \{N_1, N_2\}$. Then for $n > N$ we have

$$\begin{aligned} |(s_n + t_n) - (s + t)| &= |(s_n - s) + (t_n - t)| < |s_n - s| + |t_n - t| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $s_n + t_n \rightarrow s + t$.



9.4 Theorem Suppose $s_n \rightarrow s$ and $t_n \rightarrow t$, with $s, t \in \mathbb{R}$.
Then $s_n t_n \rightarrow st$.

This proof is quite a bit harder. We will use the fact that Thm 9.1 says convergent sequences are bounded.

Pf Let $\varepsilon > 0$.

By Thm 9.1 $\exists M > 0$ s.t. $|s_n| \leq M \quad \forall n \in \mathbb{N}$.

$\exists N_1 \in \mathbb{N}$ s.t. $n > N_1 \Rightarrow |t_n - t| < \frac{\varepsilon}{2M}$.

$\exists N_2 \in \mathbb{N}$ s.t. $n > N_2 \Rightarrow |s_n - s| < \frac{\varepsilon}{2(|t|+1)}$.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st|$$

$$\leq |s_n t_n - s_n t| + |s_n t - st|$$
$$= |s_n| |t_n - t| + |t| |s_n - s|$$

$$< M \frac{\varepsilon}{2M} + |t| \frac{\varepsilon}{2(|t|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\text{since } \frac{|t|}{2(|t|+1)} < 1.$$

9.7 Theorem

(a) $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ for $p > 0$.

As the text notes n^p has not been defined for $p \notin \mathbb{Q}$.

So, I'll only work with $p \in \mathbb{Q}$. We will need to use that $0 < a < b \Rightarrow a^p < b^p$ for $p \in \mathbb{Q} \cap (0, \infty)$.

We have only shown this for $p \in \mathbb{N}$ and $p = \frac{1}{2}$.

To ~~do~~ prove this in general, let $p = \frac{1}{k}$, $k \in \mathbb{N}$.

Suppose, $b^{\frac{1}{k}} \leq a^{\frac{1}{k}}$. Then $(b^{\frac{1}{k}})^k \leq (a^{\frac{1}{k}})^k \Rightarrow b \leq a$, which is false. Let $p = \frac{m}{k}$. Then

$$0 < a < b \Rightarrow a^m < b^m \Rightarrow a^{\frac{m}{k}} < b^{\frac{m}{k}}.$$

Now, the proof in the textbook (pg 48) is valid for $p > 0$, $p \in \mathbb{Q}$. ~~So~~ You can read it.

9.7 Theorem (9) $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1.$

This will be used in several places later in this course.

The proof uses the Binomial Theorem. This is an exercise on page 6 in Ross's textbook. You may have seen the proof in MATH 349. It says

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

$$\text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Pf of 9.7 (9). Since $n \geq 1$, $n^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$. Let $s_n = n^{\frac{1}{n}} - 1$.

Then $s_n \geq 0$. Notice $n = (1+s_n)^n$. ~~For $n \geq 2$~~
we have

$$(1+s_n)^n = \binom{n}{0} 1^n + \binom{n}{1} s_n + \binom{n}{2} s_n^2 + \dots + \binom{n}{n} s_n^n = 1 + n s_n + \frac{n(n-1)}{2} s_n^2 + \dots + s_n^n.$$

Assume $n \geq 2$. Then,

$$(1+s_n)^n \geq 1 + n s_n + \frac{n(n-1)}{2} s_n^2 \geq \frac{n(n-1)}{2} s_n^2.$$

$$\text{Thus } n > \frac{n(n-1)}{2} s_n^2.$$

$$\Rightarrow \frac{2}{n-1} > s_n^2, \Rightarrow s_n < \sqrt{\frac{2}{n-1}}.$$

$$\lim_{n \rightarrow \infty} \sqrt{\frac{2}{n-1}} = 0. \text{ Thus, } \lim_{n \rightarrow \infty} s_n = 0. \text{ Thus } \lim_{n \rightarrow \infty} n^{\frac{1}{n}} - 1 = 0$$

$$\text{So } \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1.$$

