Elementary Calculus from an Advanced Viewpoint, by Thomas, Moulton, and Zelinka

Theorem 10–3. Let f be a function, with domain $a < x < b$, whose derivative exists and is positive on that domain. Then f has an inverse, g, and if $y = f(x)$, then

$$g'(y) = \frac{1}{f'(x)}, \qquad \text{for} \qquad a < x < b. \tag{2}$$

Or, in another notation, if $y = f(x)$, then $x = g(y)$ and

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Proof. Because f' is positive on D_f, f is one-to-one from its domain to its range R_f. Thus the rule

$$g(y) = x \qquad \text{if and only if} \qquad y = f(x), \quad x \in D_f$$

defines a function g whose domain D_g is the range of f:

$$D_g = R_f.$$

Figure 10–3 will help us to follow the remaining steps in the proof. Fix $y \in D_g$ and $x = g(y) \in D_f$. Since the domain of f is (by hypothesis) the open interval $a < x < b$, there exists a positive number h such that the closed interval $[x - h, x + h]$ is in the domain of f. Let

$$y - k_1 = f(x - h), \qquad y + k_2 = f(x + h).$$

Then k_1 and k_2 are positive numbers, so y is an inner point of the domain of g, and that domain contains the closed interval $[y - k, y + k]$, where

$k = \min(k_1, k_2)$. For each $\Delta y \neq 0$, such that $|\Delta y| < k$, the intermediate-value theorem applied to f shows that there exists $\Delta x \neq 0$, such that $|\Delta x| < h$, and

$$f(x + \Delta x) = y + \Delta y, \qquad g(y + \Delta y) = x + \Delta x.$$

To prove that $g'(y)$ exists, we must show that the difference quotient

$$\frac{g(y + \Delta y) - g(y)}{\Delta y}$$

has a limit as $\Delta y \to 0$. But this is easy, because

$$g(y) = x, \qquad g(y + \Delta y) = x + \Delta x,$$
$$y = f(x), \qquad y + \Delta y = f(x + \Delta x),$$

so that

$$\frac{g(y + \Delta y) - g(y)}{\Delta y} = \frac{(x + \Delta x) - x}{f(x + \Delta x) - f(x)} = \frac{\Delta x}{f(x + \Delta x) - f(x)} \tag{3}$$

and

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x). \tag{4a}$$

By hypothesis, $f'(x) \neq 0$. Therefore, taking reciprocals in Eq. (4a) we get

$$\lim_{\Delta x \to 0} \frac{\Delta x}{f(x + \Delta x) - f(x)} = \frac{1}{f'(x)}. \tag{4b}$$

Since f is continuous on D_f and g is continuous on D_g, $\Delta x \to 0$ when $\Delta y \to 0$, and conversely. Therefore, from Eqs. (3) and (4b) we get

$$\lim_{\Delta y \to 0} \frac{g(y + \Delta y) - g(y)}{\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{f(x + \Delta x) - f(x)},$$

or

$$g'(y) = \frac{1}{f'(x)}.$$

Q.E.D.