The Rational Zeros Theorem
Section 2 in Ross Textbook

Theorem. (The Rational Zeros Theorem) [On page 9 in textbook.] Suppose $\frac{p}{q} \in \mathbb{Q}$ is a solution to

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $a_n, \ldots, a_0$ are in $\mathbb{Z}$. Assume $\frac{p}{q}$ is in reduced form. Then $p|a_0$ and $q|a_n$. (The $|$ means divides.)

Example. The roots of $21x^2 + x - 10$ are $-\frac{2}{3}$ and $\frac{5}{7}$. Notice 3 and 7 divide 21 and 2 and 5 divide 10.

Proof. Our proof will use two facts from number theory. Let m, n and p be integers.

- If m and n have no common prime factors and $p \geq 1$, then m and n^p have no common prime factors.
- If m and n have no common prime factors and $m|np$ then $m|p$.

Now to the proof. Assume

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

Then

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0. \quad (*)$$

Thus,

$$a_n p^n = -q(a_{n-1}p^{n-1}q^0 + \cdots + a_1 p q^{n-2} + a_0 q^{n-1}).$$

Thus, $q|a_n p^n$. Since q and p have no common prime factors, q and p^n have no common prime factors. Hence $q|a_n$.

But, from $(*)$ we can also derive that

$$a_0 q^n = -p(a_n p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1 p^0 q^{n-1}).$$

Thus, $p|a_0 q^n$, which implies $p|a_o$. □

Example. Consider $2x^3 + 3x^2 + 10x + 15$. Any rational root must be from the set

$$\{\pm 1, \pm 3, \pm 5, \pm 15, \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \pm\tfrac{15}{2}\}.$$

You can check that

$$2x^3 + 3x^2 + 10x + 15 = (2x+3)(x^2+5).$$

Hence, the only rational root is $-\frac{3}{2}$.

Application. We shall use the Rational Zeros Theorem to prove that certain entities cannot be rational numbers.

Example. Prove that $\sqrt{5}$ is not a rational number. *Proof.* It is a root of $x^2 - 5$. The only possible rational roots of $x^2 - 5$ are ± 1 and ± 5. But, you can check that these are not roots. Hence, $x^2 - 5$ does not have any rational roots. Hence, $\sqrt{5}$ is not rational. Note: we have not shown that $\sqrt{5}$ exist in $\mathbb{R}$. This is done in MATH 452.

Example. Let $r = \sqrt{2 + \sqrt[3]{3}}$. We show that r is not rational.

Proof. By the commonly understood definitions of square roots and cube roots[1] it is clear that

$$(r^2 - 2)^3 - 3 = 0.$$

Expanding this gives

$$r^6 - 6r^4 + 12r^2 - 8 - 3 = 0,$$

or

$$r^6 - 6r^4 + 12r^2 - 11 = 0.$$

The only possible rational roots are ± 1 and ± 11.

We know $r > 0$. Thus, if r is rational it is 1 or 11. You can check that neither of these are roots. Hence r is not rational.

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[1]The $\sqrt{x}$ is a positive real number whose square equals x, if such a real number exists. The $\sqrt[3]{x}$ is a real number whose cube equals x, if such a real number exists.