Corollary 32.10 Let f be a bdd func. on $[a,b]$. Assume $\int_a^b f$ exists. Let (P_n) be a seq. of partitions of $[a,b]$ s.t.

$$\lim_{n\to\infty} \operatorname{mesh} P = 0.$$

Then $\lim_{n\to\infty} R(f, P_n, S_n) \to \int_a^b f$, where each S_n is a sample set for P_n.

Pf Let $\varepsilon > 0$. Let $\delta > 0$ be s.t. $\left| R(f, P, S) - \int_a^b f \right| < \varepsilon$ whenever $\operatorname{mesh} P < \delta$.

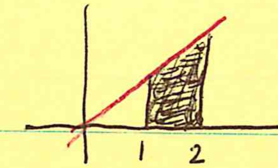
Choose N s.t. $n > N \Rightarrow \operatorname{mesh} P_n < \delta$.

Then $\left| R(f, P_n, S_n) - \int_a^b f \right| < \varepsilon$ for $n > N$.

Thus, $\lim_{n\to\infty} R(f, P_n, S_n) = \int_a^b f$. $\square$

Note In the next section we will show that all continuous functions on $[a,b]$ are integrable. We assume this for the next examples.

Example Compute $\int_1^2 x\,dx$.



Solution Since x is cont. this integral exists. We will use Cor. 32.10.

Let $P_n = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots, 1+\frac{n}{n}\}$.

Let $S_n = \{1, 1+\frac{1}{n}, \dots, 1+\frac{n-1}{n}\}$ (the left end points)

$$\text{Let } R_n = R(X, P_n, S_n) = \sum_{k=0}^{n-1} \left(1+\frac{k}{n}\right)\frac{1}{n}$$

$$= \frac{1}{n}\left(n + \frac{1}{n}\sum_{k=0}^{n-1} k\right)$$

$$= 1 + \frac{1}{n^2}\left(\frac{(n-1)(n-2)}{2}\right)$$

$$= 1 + \frac{n^2-3n+2}{2n^2}.$$

Thus $R_n \longrightarrow 1+\frac{1}{2} = 3/2$.

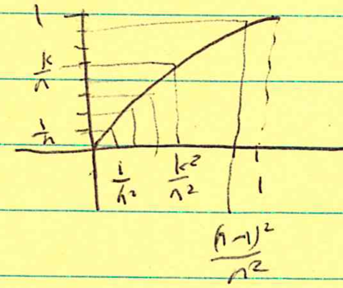
By Cor. 32.10 $\int_a^b x\,dx = \frac{3}{2}$.

Example Compute $\int_0^1 \sqrt{x}\,dx$.

Solution $\sqrt{x}$ is cont. so Cor. 32.10 can be used.

Let $P_n = \left\{0, \frac{1}{n^2}, \frac{4}{n^2}, \frac{9}{n^2}, \ldots, \frac{(n-1)^2}{n}, 1\right\}$

Let S_n = left end points.



$$\text{mesh } P_n = \max\left\{\frac{k^2-(k-1)^2}{n^2},\ k=1,\ldots,n\right\}$$

$$= \frac{n^2-(n-1)^2}{n^2} = 1 - \frac{(n-1)^2}{n^2}$$

check this!

$$\lim_{n\to\infty} \text{mesh } P_n = \lim_{n\to\infty} 1 - \frac{(n-1)^2}{n^2} = 0.$$

By Cor 32.10 $R(\sqrt{x}, P_n, S_n) \to \int_0^1 \sqrt{x}\,dx$.

$$R(\sqrt{x}, P_n, S_n) = \sum_{k=0}^{n-1} \sqrt{\frac{k^2}{n^2}}\ \frac{(k+1)^2-k^2}{n^2} = \frac{1}{n^3}\sum_{k=0}^{\infty} k(2k+1)$$

$$= \frac{1}{n^3}\left(2\sum_{k=0}^{n+1} k^2 + \sum_{k=0}^{n-1} k\right) = \frac{1}{n^3}\left[2\,\frac{(n-1)n(2(n-1)+1)}{6} + \frac{(n-1)(n-2)}{2}\right]$$

$$= \frac{4n^3-3n^2-7n+6}{6n^3} = \frac{2}{3} - \frac{1}{2n} - \frac{7}{6n^2} + \frac{1}{n^3} \longrightarrow \frac{2}{3}.$$

Thus, $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}$.