§34 FTC

FTC I

Thm 34.1 Let $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$. Assume that $g'$ is integrable over $[a,b]$. Then

$$\int_a^b g'(x)\,dx = g(b) - g(a).$$

Pf Since we are given that the integral exists we can apply the Cauchy criterion for Darboux integrability (Thm 32.5) Let $\varepsilon > 0$. $\exists$ a partition $P = \{t_0, t_1, \dots, t_n\}$ of $[a,b]$ s.t.

$$U(g', P) - L(g', P) < \varepsilon.$$

Next we apply the MVT to each interval $[t_{k-1}, t_k]$ to show $\exists\ x_k \in (t_{k-1}, t_k)$ s.t.

$$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}$$

or $$g(t_k) - g(t_{k-1}) = g'(x_k)(t_k - t_{k-1}).$$

Now,

$$g(b) - g(a) = \sum_{k=1}^{n} \left[ g(t_k) - g(t_{k-1}) \right] \quad \text{(telescoping series)}$$

$$= \sum_{k=n}^{n} g'(x_k)(t_k - t_{k-1})$$

Since, $m(g', [t_{k-1}, t_k]) \leq g'(x_k) \leq M(g', [t_{k-1}, t_k])$

it follows that

$$L(g', P) \leq g(b) - g(a) \leq U(g', P).$$

But we also know

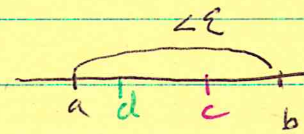
$$L^*(g', P) \leq \int_a^b g'(x)\,dx \leq U^*(g', P).$$

Therefore,* $$\left| \int_a^b g'(x)\,d(x) - \big(g(b) - g(a)\big) \right| < \varepsilon.$$

Since this holds for any $\varepsilon > 0$ it must be that

$$\int_a^b g'(x)\,dx = g(b) - g(a). \quad \square$$

* <u>Lemma</u>: Suppose $a < c < b$, $a < d < b$ and $b - a < \varepsilon$ $(\varepsilon > 0)$.
Then $|c - d| < \varepsilon$.




<u>Pf</u>

$$c - d < b - d$$
$$a - b < d - b$$
$$-\varepsilon < a - b < d - b < d - c$$
$$-\varepsilon < d - c$$
$$c - d < \varepsilon \quad (1).$$

$$d - c < b - c$$
$$a - b < c - b$$
$$-\varepsilon < a - b < c - b < c - d$$
$$-\varepsilon < c - d \quad (2).$$

$(1) \,\&\, (2) \Rightarrow |c - d| < \varepsilon$

$\square$

# FTC II

**Thm 34.3** Let $f$ be continuous on $[a,b]$. For $x \in [a,b]$ define

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F(x)$ is continuous on $[a,b]$ and $F'(x) = f(x)$, $\forall x \in (a,b)$.

**Pf** Let $B > 0$ be a bound for $f$, that is $|f(x)| \le B$ $\forall x \in [a,b]$.

We will show that $F(x)$ is uniformly cont. on $[a,b]$. This part of the proof does not assume $f$ is cont., merely integrable on $[a,b]$.

Let $\varepsilon > 0$. For $x, y \in [a,b]$ with $x < y$ and $|x-y| < \frac{\varepsilon}{B}$ we have

$$|F(x) - F(y)| = \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt \le \int_x^y B\,dx = B(y-x) < \varepsilon.$$

$$\parallel$$

$$\left| \int_a^x F(x)\,dx - \int_a^y F(x)\,dx \right| = \left| \int_a^y F + \int_y^x F - \int_y^x \right|$$

Thus, $F(x)$ is uniformly cont. on $[a,b]$.

Let $x_0 \in (a,b)$. For any $x \neq x_0$ in $[a,b]$ we have

$$\int_{x_0}^{x} f(t) = F(x) - F(x_0) \qquad \left( \int_a^b f = -\int_b^a f \text{ by definition} \right)$$

by 34.1. Hence, $\dfrac{F(x)-F(x_0)}{x-x_0} = \dfrac{1}{x-x_0}\displaystyle\int_{x_0}^{x} f(t)\,dt.$

Since $x_0$ is fixed, $f(x_0)$ is fixed and

$$\int_{x_0}^{x} f(x_0)\,dt = f(x_0)\int_{x_0}^{x} 1\,dt = f(x_0)(x-x_0)$$

Thus, $$f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)\,dt.$$

Hence, $$\frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x} f(t)\,dt - f(x_0)$$

$$= \frac{1}{x-x_0}\int_{x_0}^{x} f(t) - f(x_0)\,dt$$

Now let $\varepsilon > 0$. Since <u>f is cont.</u> a $x_0$ $\exists\, \delta > 0$ s.t.

$$t \in (a,b),\ |t-x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon.$$

Thus, $$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| \leq \frac{1}{x-x_0}\int_{x_0}^{x} |f(t)-f(x_0)|\,dt$$

$$< \frac{1}{x-x_0}\,\varepsilon(x-x_0) = \varepsilon$$

provided $|x-x_0| < \delta$, for $x > x_0$.

If $x_0 < x$ it is only slightly different.

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq \left| \frac{1}{x - x_0} \right| \left| \int_{x_0}^{x} f(t) - f(x_0)\, dt \right|$$

$$= \frac{1}{x_0 - x} \left| \int_{x}^{x_0} f(t) - f(x_0)\, dt \right|$$

$$\leq \frac{1}{x_0 - x} \int_{x}^{x_0} |f(t) - f(x_0)|\, dt$$

$$< \frac{1}{x_0 - x} \varepsilon (x_0 - x) = \varepsilon.$$

Either way $|x - x_0| < \delta \Rightarrow$

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon$$

for $x \in (a,b)$. Therefore, by definition,

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

Thus, $F'(x_0) = f(x_0)$. $\square$