§34 continued. This section developes two integration techniques, Substitution (34.4) and integration by parts (34.2).

34.4 Let D and R be open intervals and let $u: D \to R$. Assume u is differentiable and that u' is cont. Assume $f: R \to \mathbb{R}$ is cont. Thus $f \circ u$ is ~~cont~~. defined and cont. as a function on D. Then

$$\int_a^b f \circ u(x)\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

for a, b in D.

Pf Let $c \in D$. Let $F(u) = \int_c^u f(t)\,dt$. By FTC II

$$F'(u) = f(u) \quad \forall\, u \in R.$$

Let $g = F \circ u$. Then

$$g'(x) = F'(u(x))\, u'(x) = f(u(x))\, u'(x).$$

By FTC I

$$\int_a^b f \circ u(x) \cdot u'(x)\, dx = \int_a^b g'(x)\, dx = g(b) - g(a)$$

$$= F(u(b)) - F(u(a)) = \int_c^{u(b)} f(t)\, dt - \int_c^{u(a)} f(t)\, dt$$

$$= \int_{u(a)}^{u(b)} f(t)\, dt. \qquad \square$$

In other words, substitution is the chain rule backward. So too integration by parts is the product rule backwards.

34.2 Let u and v be cont. functions on $[a,b]$. Assume they are diff. on (a,b) and that u' and v' are integrable on $[a,b]$. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)\,v(x)\,dx.$$

Pf Let $g = uv$. Then $g' = uv' + u'v$. By FTC I

$$\int_a^b g'(x)\,dx = g(b) - g(a) = u(b)v(b) - u(a)v(a) = u(x)v(x)\Big|_a^b.$$

Thus

$$\int_a^b uv'\,dx + \int_a^b u'v\,dx = u(x)\,v(x)\Big|_a^b$$

or

$$\int_a^b uv'\,dx = u(x)\,v(x)\Big|_a^b - \int_a^b u'v\,dx. \quad \boxed{\circ}$$