# Cauchy Sequences

Section 10 of the textbook proves two important theorems, The Monotone Convergence Theorem (10.2) and The Cauchy Convergence Theorem (10.11), and introduces the concepts of $\limsup$ and $\liminf$ used in the proof of 10.11.

**Theorem.** (10.2) Let $(a_n)$ be a sequence of real numbers. If it is bounded and monotone it converges to a finite limit.

*Proof.* There are two cases.

**Case 1.** Suppose $(a_n)$ is decreasing or nonincreasing. Since the underlying set is bounded it is bounded below and thus has a glb. Call it $L$. We will show that

$$\lim_{n\to\infty} a_n = L.$$

Let $\epsilon > 0$. There exists an natural number $N$ such that

$$a_N < L + \epsilon$$

since otherwise $L + \epsilon$ would be a lower bound for $\{a_n\}$ that is greater than $L$. For all $n > N$ we know $L \leq a_n \leq a_N$. Thus,

$$L - \epsilon < a_n < L + \epsilon$$

for all $n > N$. Hence, $|a_n - L| < \epsilon$ when $n > N$. Thus, $s_n \to L$ as claimed.

**Case 2.** Suppose $(a_n)$ is increasing or nondecreasing. See textbook.

$\square$

**Note.** Theorem 10.2 fails in $\mathbb{Q}$.

**Definition.** A sequence $(a_n)$ is **Cauchy** if for every $\epsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $m, n > N$ implies $|a_m - a_n| < \epsilon$.

**Theorem.** (10.11) A sequence of real numbers $(a_n)$ converges to a finite limit iff it is Cauchy.

The proof is broken down into three parts. The third part will require the introduction of a new idea: lim sup and lim inf. We will do the first two parts, then pause to develop lim sup and lim inf, before coming back and doing the last part of the proof of 10.11.

**Part I (Lemma 10.9).** Convergent sequences are Cauchy.

*Proof.* Suppose $(a_n)$ converges to $L$. Let $\epsilon > 0$.

$\exists N$ s.t. $n > N$ implies $|a_n - L| < \epsilon/2$.

Now suppose $m, n > N$. Then
$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \epsilon/2 + \epsilon/2 = \epsilon$.
Thus, $(a_n)$ is Cauchy. $\square$

**Note.** Lemma 10.9 is valid for $\mathbb{Q}$.

**Part II (Lemma 10.10).** Cauchy sequences are bounded.

*Proof.* Let $(a_n)$ be a Cauchy sequence. Let $\epsilon = 1$. Let $N$ be such that if $m, n > N$ then

$$|a_m - a_n| < 1.$$

It follows that if $n > N$ then $|a_n - a_{N+1}| < 1$. Thus, $|a_n| < |a_{N+1}| + 1$.

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**Note:** That last claim uses a result from Exercise 3.5 that for any two real numbers $a$ and $b$

$$\big||a| - |b|\big| \leq |a - b|.$$

In our case this becomes

$$\big||a_n| - |a_{N+1}|\big| \leq |a_n - a_{N+1}| < 1.$$

Thus,

$$-1 < |a_n| - |a_{N+1}| < 1.$$

Ergo, $|a_n| < |a_{N+1}| + 1$.

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Now we have a bound when $n > N$. It could be one of the earlier terms was even larger so we do the following. Let

$$M = \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_{N+1}| + 1\}.$$

Then $M$ is a bound for the sequence. $\square$

Now we pause to develop a new tool to analyzing sequences.

**Definition.** Let $(a_n)$ be a sequence. Then

$$\limsup a_n = \lim_{N\to\infty} \sup\{a_n \,|\, n > N\}$$

and

$$\liminf a_n = \lim_{N\to\infty} \inf\{a_n \,|\, n > N\}$$

**Examples.** I'll do some examples in class to make these understandable. You have some examples in your homework as well.

Let $V_N = \sup\{s_n \,|\, n > N\}$ and $U_N = \inf\{s_n \,|\, n > N\}$. Since $V_N \geq U_N$ we have $\limsup a_n \geq \liminf a_n$.

Notice that $V_N$ is nonincreasing and $U_N$ is nondecreasing. (See also Exercise 4.7 on page 27.) Thus, if the sequence is bounded lim sup and lim inf will be finite real numbers.

Thus, $\limsup s_n \leq V_N$ and $\liminf s_n \geq U_N$ for any $N \in \mathbb{N}$.

These observations will be useful.

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**Definition.** Let $(a_n)$ be a sequence. Then $c \in \mathbb{R}$ is a **cluster point** of $(a_n)$ if $\forall\, \epsilon > 0$ and $\forall N \in \mathbb{N} \;\; \exists n > N$ such that

$$|a_n - c| < \epsilon.$$

You can check that if $a_n \to L \in \mathbb{R}$, then $L$ is a cluster point.

**Theorem.** **(10.7)** Let $(s_n)$ be a sequence and $L \in \mathbb{R}$. If $\limsup s_n = \liminf s_n = L$ then $\lim_{n\to\infty} s_n = L$.

*Proof.* Pick any $\epsilon > 0$.

Since, $V_N \to L$ we know $\exists\, N_1$ s.t. $|L - V_{N_1}| < \epsilon$.

Thus,

$$-\epsilon < L - V_{N_1} < \epsilon.$$

Thus,

$$V_{N_1} - \epsilon < L < V_{N_1} + \epsilon.$$

Thus,

$$V_{N_1} < L + \epsilon.$$

Thus,

$$\sup\{s_n \,|\, n > N_1\} < L + \epsilon.$$

Thus,

$$s_n < L + \epsilon, \ \ \forall n > N_1.$$

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Since $U_N \to L$ we know $\exists\, N_2$ s.t. $|L - U_{N_2}| < \epsilon$.

Thus,

$$-\epsilon < L - U_{N_2} < \epsilon.$$

Thus,

$$U_{N_2} - \epsilon < L < U_{N_2} + \epsilon.$$

Thus,

$$L - \epsilon < U_{N_2}.$$

Thus,

$$L - \epsilon < \inf\{s_n \,|\, n > N_2\}.$$

Thus,

$$L - \epsilon < s_n, \ \ \forall n > N_2.$$

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Let $N = \max\{N_1, N_2\}$. Now for all $n > N$ we have

$$L - \epsilon < s_n < L + \epsilon.$$

Thus,

$$-\epsilon < s_n - L < \epsilon.$$

Thus,

$$|s_n - L| < \epsilon.$$

Hence, $\lim_{n\to\infty} s_n = L$. $\square$

**Part III.** Cauchy sequences converge.

The book's proof uses a fact often called "the $\epsilon$-principle": If $a \leq b + \epsilon$ for all $\epsilon > 0$ then $a \leq b$. *Proof:* Suppose $a > b$. Let $\epsilon = (a-b)/2 > 0$. Now $a \leq b + (a-b)/2$ implies $a \leq b$. Contradiction!

*Proof.* Let $(s_n)$ be a Cauchy sequence. By Lemma 10.10 $\limsup s_n$ and $\liminf s_n$ are finite. We will show

$$\limsup s_n = \liminf s_n.$$

Then by Theorem 10.7 we are done. Notice that from the definition

$$\limsup s_n \geq \liminf s_n$$

is always true. So, we only need to show that $\limsup s_n \leq \liminf s_n$.

Let $\epsilon > 0$.

Then $\exists\, N \in \mathbb{N}$ s.t. $m, n > N$ implies $|s_n - s_m| < \epsilon$.

Thus, $s_n < s_m + \epsilon, \quad \forall n > N$ and any fixed $m > N$.

Thus, $s_m + \epsilon$ is an upper bound of $\{s_n \mid n > N\}$.

Thus, $V_N \leq s_m + \epsilon$, $\forall m > N$.

Thus, $V_N - \epsilon$ is a lower bound of $\{s_m \mid m > N\}$.

Thus, $V_N - \epsilon \leq \inf\{s_m \mid m > N\} = U_N$.

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Now, $\limsup s_n \leq V_N \leq U_N + \epsilon \leq \liminf s_n + \epsilon$.

Thus, $\limsup s_n \leq \liminf s_n + \epsilon$ for all $\epsilon > 0$. By the $\epsilon$-principle we are done! $\square$