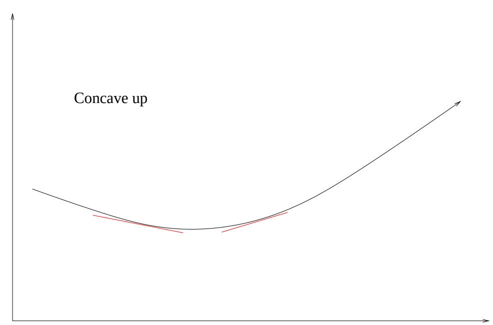
Concavity

Concavity Test (Stewart's Calculus Textbook, Appendix D)

Definition. Let f be a differentiable function on the open interval I. Let $T_a(x) = f'(a)(x-a) + f(a)$, $\forall\, a \in I$. This is the tangent line to f at $x = a$.

(a) If $\forall a \in I$, $f(x) > T_a(x)$ $\forall x \in I - \{a\}$, then we say f is concave up.

(b) If $\forall a \in I$, $f(x) < T_a(x)$ $\forall x \in I - \{a\}$, then we say f is concave down.



Theorem. (a) If $f''(x) > 0\ \forall\, x \in I$, then f is concave up on I.
(b) If $f''(x) < 0\ \forall\, x \in I$, then f is concave down on I.

Proof of (a). Let $a \in I$. For now suppose $x > a$, $x \in I$. Apply the MVT to f over $[a, x]$ to get a value $c \in (a, x)$ s.t.

$$f(x) - f(a) = f'(c)(x-a).$$

Since, $f'' > 0$ on I, f' is increasing. Thus,

$$f'(a) < f'(c).$$

The rest is just algebra.

$$f'(a)(x-a) + f(a) < f'(c)(x-a) + f(a) = f(x).$$

Thus,

$$f(x) > f'(a)(x-a) + f(a) = T_a(x),$$

for $x > a$.

When $x < a$ the argument is similar. The only differences are that $f'(a)$ will be greater than $f'(c)$, but $x - a$ will be negative, which will flip the inequality to give the desired result. You should work this out. $\square$