

Ordered Fields

Definition. Let S be a set. A **binary operation** is a map from $S \times S$ to S .

Definition. An **ordered field** is a set $\mathbb{F}$ with more than one member together with two binary operations, addition $+$ and multiplication $\cdot$ and an order relation $\leq$ satisfying the axioms below for all a , b , and c in $\mathbb{F}$. (The $\cdot$ is often left unwritten.)

A1. $a + (b + c) = (a + b) + c$.

A2. $a + b = b + a$.

A3. $\exists 0 \in \mathbb{F}$ such that $a + 0 = a$.

A4. $\exists -a \in \mathbb{F}$ such that $a + (-a) = 0$.

M1. $a(bc) = (ab)c$.

M2. $ab = ba$.

M3. $\exists 1 \in \mathbb{F}$ such that $a \cdot 1 = a$.

M4. If $a \neq 0$, $\exists a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$.

DL. $a(b + c) = ab + ac$.

O1. Either $a \leq b$ or $b \leq a$ is true.

O2. If $a \leq b$ and $b \leq a$, then $a = b$.

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. If $a \leq b$, then $a + c \leq b + c$.

O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Note: Given $\leq$ then $\geq$, $<$ and $>$ are defined as usual.

From these beginnings the following can be proven.

Theorem 3.1 Let $\mathbb{F}$ be a field. Then $\forall a, b, c$ in $\mathbb{F}$ the following hold.

- (i) If $a + c = b + c$, then $a = b$.
- (ii) $a \cdot 0 = 0$.
- (iii) $(-a)b = -(ab)$.
- (iv) $(-a)(-b) = ab$.
- (v) If $ac = bc$ and $c \neq 0$, then $a = b$.
- (vi) If $ab = 0$ then either $a = 0$ or $b = 0$.

Theorem 3.2 Let $\mathbb{F}$ be an ordered field. Then $\forall a, b, c$ in $\mathbb{F}$ the following hold.

- (i) If $a \leq b$, then $-b \leq -a$.
- (ii) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.
- (iii) If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.
- (iv) $0 \leq a^2$.
- (v) $0 < 1$.
- (vi) If $0 < a$, then $0 < a^{-1}$.
- (vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Some Proofs.

3.1(i) If $a + c = b + c$, then $a = b$.

$$\begin{array}{ll}
 a + c & = b + c & \text{is given.} \\
 (a + c) + (-c) & = (b + c) + (-c) & \text{by def. of binary op.} \\
 a + (c + (-c)) & = b + (c + (-c)) & \text{by axiom A1 applied to both sides.} \\
 a + 0 & = b + 0 & \text{by A4.} \\
 a & = b & \text{by A3.}
 \end{array}$$

3.1(ii) $a \cdot 0 = 0$.

$$\begin{array}{ll}
 a \cdot 0 & = a(0 + 0) & \text{A3} \\
 a \cdot 0 & = a \cdot 0 + a \cdot 0 & \text{DL} \\
 a \cdot 0 + 0 & = a \cdot 0 + a \cdot 0 & \text{A3} \\
 0 + a \cdot 0 & = a \cdot 0 + a \cdot 0 & \text{A2} \\
 0 & = a \cdot 0 & \text{3.1(i)}
 \end{array}$$

3.1(iii) $(-a)b = -(ab)$.

$$\begin{array}{ll}
 0 \cdot b & = 0 & \text{3.1(ii)} \\
 (a + (-a))b & = 0 & \text{A4} \\
 ab + (-a)b & = 0 & \text{DL} \\
 ab + -(ab) & = 0 & \text{A4} \\
 ab + -(ab) & = ab + (-a)b & \text{since } 0=0 \\
 -(ab) & = (-a)b & \text{A2 \& 3.1(i)}
 \end{array}$$

3.1(iv) $(-a)(-b) = ab$.

It will be useful to first prove that $-(-x) = x$. Since $(-x) + x = 0$ and $(-x) + (-(-x)) = 0$, Theorem 3.1(i) implies that $-(-x) = x$.

$$\begin{array}{ll}
 (-a)(-b) & = -(a(-b)) & \text{3.1(iii)} \\
 & = -((-b)a) & \text{M2} \\
 & = -(-(ba)) & \text{3.1(iii)} \\
 & = ab
 \end{array}$$

3.1(v) $ac = bc \ \& \ c \neq 0 \implies a = b$.

First, c^{-1} exists by M4.

$$\begin{array}{rcl} (ac)c^{-1} & = & (bc)c^{-1} \\ a(cc^{-1}) & = & b(cc^{-1}) \quad \text{M1} \\ a \cdot 1 & = & b \cdot 1 \quad \text{M4} \\ a & = & b \quad \text{M3} \end{array}$$

3.1(vi) $ab = 0 \implies a = 0 \text{ or } b = 0$.

Let $ab = 0$ and suppose $a \neq 0$ and $b \neq 0$. Thus, a^{-1} and b^{-1} exist.

$$\begin{array}{rcl} ab(b^{-1}) & = & 0 \cdot b^{-1} \\ a(bb^{-1}) & = & 0 \quad \text{M1 \& 3.1(ii)} \\ a \cdot 1 & = & 0 \quad \text{M4} \\ a & = & 0 \quad \text{M3} \end{array}$$

But $a = 0$ contradicts our supposition. Thus, if $ab = 0$ then $a = 0$ or $b = 0$.

3.2(i) $a \leq b \implies -b \leq -a$.

$$\begin{array}{rcl} a & \leq & b \quad \text{given} \\ a + ((-a) + (-b)) & \leq & b + ((-a) + (-b)) \quad \text{O4} \\ (a + (-a)) + (-b) & \leq & b + ((-b) + (-a)) \quad \text{A1 \& A2} \\ 0 + (-b) & \leq & (b + (-b)) + (-a) \quad \text{A4 \& A1} \\ -b & \leq & 0 + (-a) \quad \text{A3 \& A4} \\ -b & \leq & -a \quad \text{A3} \end{array}$$

3.2(ii) $a \leq b \ \& \ c \leq 0 \implies bc \leq ac$.

I'll need to use that $-0 = 0$ so I'll prove that first.

$$\begin{array}{lll} (i) & 0 + (-0) = 0 & \text{by A4.} \\ (ii) & 0 = 0 + 0 & \text{by A3.} \\ (i)\&(ii) & \implies 0 + (-0) = 0 + 0. \\ \text{Thus,} & -0 = 0 & \text{by Thm 3.1(i).} \end{array}$$

Assume $a \leq b$ and $c \leq 0$. Since $c \leq 0$, 3.2(i) implies $-0 \leq -c$. Thus, $0 \leq -c$.

$$\begin{array}{lll} a(-c) & \leq & b(-c) \quad \text{O5} \\ -(ac) & \leq & -(bc) \quad \text{M2,3.1(iii),M2} \\ bc & \leq & ac \quad \text{3.2(i)} \end{array}$$

3.2(iii) $0 \leq x \ \& \ 0 \leq y \implies 0 \leq xy$.

Apply O5 with $a = 0$, $b = x$ and $c = y$.

3.2(iv) $0 \leq a^2$.

Either $a \leq 0$ or $0 \leq a$ by O1.

Suppose $0 \leq a$. Then $0 \leq a^2$ by O5 and Thm 3.1(ii).

Suppose $a \leq 0$. By 3.2(i) we have $0 \leq -a$. Thus, $0 \leq (-a)^2$.

Since $(-a)^2 = a^2$ by 3.1(iv) we have $0 \leq a^2$.